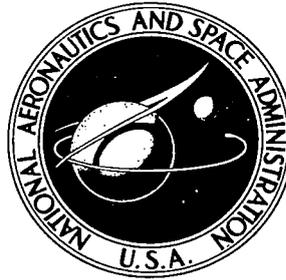


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GENERALIZED RING STIFFNESS MATRIX FOR RING-STIFFENED SHELLS OF REVOLUTION

by George E. Weeks and Joseph E. Walz

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SUMMARY

A generalized set of nonlinear boundary conditions and ring stiffness coefficients are derived for a shell of revolution with an elastic ring of arbitrary cross section attached to its boundary in an arbitrary manner. The effects of shear deformation, internal pressure, eccentricity, restraint of warping, torsion, out-of-plane bending, and prestress of the ring are included in the analysis. The boundary conditions are applicable to shells with an axisymmetric initial equilibrium state of stress and deformation which are subject to small asymmetric deflections away from the initial state due to either bifurcation buckling or the addition of an asymmetric load. The results are presented in a form which can be easily introduced into a computer program for shells of revolution.

INTRODUCTION

Shells of revolution with rings attached at the boundaries are common structural elements in aerospace vehicles. A typical example is the envelope or "aeroshell" of a spacecraft designed for entry into a planetary atmosphere. The analysis and design of such structural elements require a knowledge of the edge restraint or boundary conditions on the shell provided by an elastic ring of arbitrary cross section. However, because the stiffness characteristics of an arbitrary ring are very complex and lead to complicated boundary conditions for the shell to which the ring is attached, analytical procedures for taking into account ring behavior are usually based on approximate and many times inadequate theory.

At the time this study was initiated, the most comprehensive treatment of ring boundary conditions in the literature was given by Cohen who presented in reference 1 the boundary conditions for an arbitrary shell of revolution with an elastic ring of arbitrary cross section attached to its boundary. Effects of prestress, shear deformation, and restraint of warping in the ring were neglected. Later in reference 2, Cohen included some prestress effects, but still, his analysis is only applicable to a ring whose shear center and centroid coincide, and in addition, the normal to the shell at the point of

attachment must pass through the ring centroid. The present theory does not have these restrictions and has been developed independently of the theory of reference 1.

The purpose of this paper is (1) to present the assumptions and theoretical development of a ring theory for the nonlinear boundary conditions of a general shell of revolution with an elastic ring of arbitrary cross section attached to its boundary, (2) to present the resulting equations (ring stiffness coefficients) in a form which can be easily incorporated into a computer program for shells of revolution, and (3) to demonstrate the use of the equations with selected results which show the influence of ring flexibility and ring eccentricity. The ring theory is general and includes shear deformation, restraint of warping, out-of-plane bending, eccentricity, internal pressure, torsion, and axisymmetric prestress of the ring. (For stress analysis, ring prestress refers to any initial internal load in the ring with no external load on the shell. For stability analysis, ring prestress also includes internal loading in the ring prior to buckling due to the prebuckling deformation of the shell from applied loads.)

The ring theory is derived on the basis of the total potential energy of the ring. The energy of restraint of warping and torsion are included in an approximate manner similar to that of straight-beam theory whereas the energy of bending and extension are considered rigorously. The results are also applicable, with certain noted exceptions, to the special but important case of a pressurized ring of closed cross section.

SYMBOLS

The units used for the physical quantities defined in this paper are given both in the U.S. Customary Units and in the International System of Units (SI) (ref. 3). Conversion factors pertinent to the present analysis are presented in appendix A.

A,B,C,D,F	functions of displacement variables (see eqs. (3))
A_r	cross-sectional area of ring
\bar{A}_r	area enclosed by ring of closed cross section
a_i	geometrical expressions defined by equations (B3); $i = 1,2,3,4,5$
b_i	constants defined by equations (B5) and (C14) which relate ring variables to shell variables; $i = 1,2, \dots, 21$
E	Young's modulus of ring and shell

e_0	circumferential strain of centroidal axis of ring
e_x, e_z	thickness-compression strains at any point in ring cross section
e_{xz}, e_{yz}, e_{yx}	shear strains at any point in ring cross section
e_y	circumferential strain at any point in ring cross section
F_r, F_θ, F_a	middle-surface stress resultants at shell boundary in cylindrical coordinate system (see fig. 1)
G	shear modulus
G_{ij}	elements of 4×4 stiffness matrix
$\bar{G}_{ij,A}, \bar{G}_{ij,B}$	ring stiffness coefficients defined by equations (E1) and (E2), respectively
$\tilde{G}_{ij} = \bar{G}_{ij,A} + \bar{G}_{ij,B}$	
h	thickness of shell wall
I_x, I_z, I_{xz}	moments and product of inertia of ring cross section with respect to ring coordinate system
J	torsional constant of ring cross section
l_i	elements of 4×1 column matrix in boundary-condition equation identified by external loadings
M_x, M_z	bending moments in ring
M_η	meridional bending-moment resultant in shell
m_η	nondimensional meridional bending-moment resultant in shell, $M_\eta^{(n)} / q_n h^2$
N	hoop force in ring
N_η, Q_η	middle-surface stress resultants in shell with respect to intrinsic coordinate system (see fig. F1)

n	Fourier index
p	internal pressure in ring
p_i	geometrical expressions defined by equations (D2); $i = 1, 2, \dots, 23$
Q_i	generalized forces defined by equations (D1); $i = 1, 2, 3, 4, 5, 6$
q_n	Fourier coefficient in expansion for normal pressure on conical shell
\hat{q}	constant radial line loading
R_θ	radius of curvature of shell which generates shell circumference
r	radius of middle surface of shell at any point along shell meridian (see fig. 1)
r_0	radius of middle surface of conical shell at large end
r_s, r_c	radius of line of shear centers and line of centroids of ring cross section, respectively
S	shear-stiffness parameter, $\frac{GA_r r_c^2}{2EI_x}$
s	meridional distance measured from large end of conical shell
s_{\max}	total meridional distance of conical shell
T	extensional-stiffness parameter, $\frac{EA_r r_c^2}{EI_x}$
t_η	nondimensional meridional stress resultant, $N_\eta^{(n)} / q_n r_0$
U_f	potential energy of external forces applied to ring
U_p	potential energy of pressurizing gas
U_r	total strain energy of ring

U_1, U_2, U_3	strain energy of ring bending and extension, restraint of warping, and shear, respectively
u, v, w	axial, circumferential, and radial displacements, respectively, of shell middle surface in cylindrical coordinate system (see fig. 1)
u_a, v_a, w_a	axial, circumferential, and radial displacements, respectively, of any point in ring cross section in cylindrical coordinate system
u_p, v_p, w_p	axial, circumferential, and radial displacements, respectively, of any point in shell wall in cylindrical coordinate system (see fig. B2)
u_s, v_s, w_s	axial, circumferential, and radial displacements, respectively, of shear center of ring in cylindrical coordinate system (see fig. B1)
\tilde{u}, \tilde{w}	meridional and normal displacements, respectively, of shell middle surface in intrinsic coordinate system (see fig. F1)
w_η	nondimensional normal displacement, $\frac{Eh}{q_n r_0^2} \tilde{w}^{(n)}$
x, y, z	ring coordinate system (see fig. B1)
x_c, z_c	axial and radial distance, respectively, from ring shear center to centroid (see fig. B3)
\bar{x}, \bar{z}	axial and radial distance, respectively, from ring shear center to point of attachment of ring to shell (see fig. B3)
z_0	eccentricity of ring centroid normal to shell middle surface measured positively along outward normal
z_{sh}	normal distance from shell middle surface in intrinsic coordinate system to point of attachment of ring to shell
\tilde{z}	distance of shell displacements u_p , v_p , and w_p from shell middle surface
α	angle which meridian line of shell makes with radial direction (see fig. F1)
β	rotation of ring cross section about axis of ring shear center (see fig. B1)

Γ	warping constant of straight-beam theory
γ_{yx}, γ_{yz}	Timoshenko-type ring transverse shear strains
δ	operator which indicates variation of variable or function
ξ	angle of twist of ring cross section with respect to its shear center
η	meridional coordinate
η_i	quantities defined by equations (C3b) to (C3f); $i = 1,2,3,4,5$
θ	circumferential coordinate

$$\Lambda = \frac{F}{r_s} + \phi'$$

$$\xi = C + A'$$

ϕ rotation of shell appearing in equation (B2)

Subscripts:

A initial equilibrium state

B incremental equilibrium state

Superscript:

(n) Fourier coefficient of indicated variable

A prime indicates a derivative with respect to the circumferential coordinate θ .

ANALYSIS

Statement of the Problem

The most general boundary conditions for a shell of revolution supported by some generalized unloaded elastic medium at its boundaries can be written in the following convenient matrix form (see, for instance, ref. 1):

$$r \begin{Bmatrix} F_a \\ F_\theta \\ F_r \\ M_\eta \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ & G_{22} & G_{23} & G_{24} \\ & & G_{33} & G_{34} \\ \text{Symmetric} & & & G_{44} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \phi \end{Bmatrix} = \begin{Bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{Bmatrix} \quad (1)$$

The G_{ij} are stiffness coefficients which define the stiffness characteristics of the supporting medium; u , v , w , and ϕ are the middle-surface shell variables and F_a , F_θ , and F_r are stress resultants at the shell boundary in the axial, circumferential, and radial directions, respectively. The quantity M_η is the moment per unit length imposed on the shell boundary by the elastic support which produces a rotation about a tangent to the circumference. Here, all terms are referred to the cylindrical coordinate system shown in figure 1 and the positive sign in equations (1) applies if the ring is connected to the positive (terminal) end of the shell.

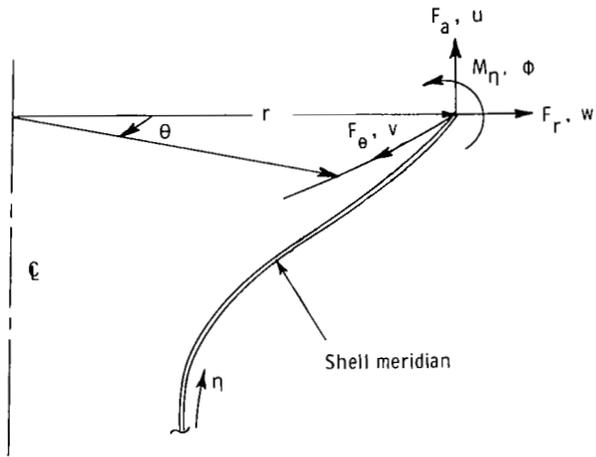


Figure 1.- Stress resultants, moment resultant, displacements, and rotation of shell at boundary.

The column vector with elements l_i is nonzero if additional external middle-surface forces on the boundary are specified.

The effects of external loads on the ring itself are not considered in the present paper. However, the method for taking these effects into account is derived in detail in reference 4.

The basic problem is to determine the stiffness coefficients G_{ij} of an elastic ring of arbitrary cross section attached in an arbitrary manner to the boundary of an arbitrary shell of revolution so that the boundary conditions (eqs. (1)) for such a shell can be defined. The procedure used to determine the desired coefficients will be based on the variation of the total potential energy of the ring.

Potential Energy of the Ring

To formulate the potential energy of the ring, the displacements of any point in the cross section of the ring are referred to displacements of the shear center. The

displacements of the shell through the thickness are then determined and the ring displacements are expressed in terms of the shell middle-surface displacements in the cylindrical coordinate system (fig. 1) by requiring compatibility of displacements and rotations at the line of attachment of the ring to the shell and by assuming that shell displacements vary linearly through the thickness of the shell. These kinematic relationships are derived in appendix B. The final expressions for the displacements of a general point in the ring cross section u_a , v_a , and w_a in terms of the shell middle-surface displacements u , v , and w ; rotation ϕ ; and ring transverse shear strains γ_{yz} and γ_{yx} are

$$\left. \begin{aligned} u_a(\theta, z) &= B(\theta) + (z - z_c)\phi(\theta) \\ v_a(\theta, x, z) &= C(\theta) - \frac{z - z_c}{r_s} D(\theta) - \frac{x - x_c}{r_s} F(\theta) \\ w_a(\theta, x) &= A(\theta) - (x - x_c)\phi(\theta) \end{aligned} \right\} \quad (2)$$

where

$$\left. \begin{aligned} A(\theta) &= -w(\theta) + b_1\phi(\theta) \\ B(\theta) &= -u(\theta) - b_2\phi(\theta) \\ C(\theta) &= b_3v(\theta) + b_5w'(\theta) + b_{10}u'(\theta) + b_7\phi'(\theta) + b_{15}\gamma_{yz}(\theta) + b_{17}\gamma_{yx}(\theta) \\ D(\theta) &= -r_s \left[b_4v(\theta) + b_6w'(\theta) + b_{11}u'(\theta) + b_8\phi'(\theta) + b_{16}\gamma_{yz}(\theta) + b_{19}\gamma_{yx}(\theta) \right] \\ F(\theta) &= -r_s \left[b_{12}u'(\theta) + b_9\phi'(\theta) + b_{18}\gamma_{yx}(\theta) \right] \end{aligned} \right\} \quad (3)$$

and where r_s is the radius of the line of shear centers of the ring cross section and the coefficients b_i are given by equations (B5) in appendix B.

The total potential energy is composed of the internal strain energy U_r of the ring, the potential energy U_f of the forces acting on the ring, and, for pressurized rings of closed cross section, the potential energy U_p of the pressurizing gas.

Strain energy.- The strain energy for a circular ring, including the energy of bending, extension, transverse shear, and the approximate energy of torsion and restraint of warping, is derived in appendix C in terms of the shell variables and the ring transverse shear strains. The result is

$$\begin{aligned}
U_r = & \frac{E}{2} \int_0^{2\pi} \left(2 \frac{Ne_0}{E} - A_r e_0^2 - \frac{2}{r_c} \frac{e_0}{E} M_x + \frac{M_x}{E} \eta_1 + \frac{M_z}{E} \eta_2 \right) r_c d\theta \\
& + \frac{E\Gamma}{2r_s r_c} \int_0^{2\pi} (b_{20}\phi'' + b_{21}u'')^2 r_c d\theta + \frac{A_r G}{2} \int_0^{2\pi} (\gamma_{yz}^2 + \gamma_{yx}^2) r_c d\theta \\
& + \frac{GJ}{2} \frac{r_s}{r_c} \int_0^{2\pi} (b_{20}\phi' + b_{21}u')^2 r_c d\theta
\end{aligned} \tag{4}$$

where r_c is the radius of the line of centroids of the ring cross section, the quantities e_0 and η_i are given by equations (C3) in appendix C, and b_{20} and b_{21} are given by equations (C14) in appendix C.

Potential energy of the external forces.- The potential energy of the external forces applied to the ring is

$$U_f = \int_0^{2\pi} \left[\left(\frac{rF_a}{2} - l_1 \right) u + \left(\frac{rF_\theta}{2} - l_2 \right) v + \left(\frac{rF_r}{2} - l_3 \right) w + \left(\frac{rM_\eta}{2} - l_4 \right) \phi \right] d\theta \tag{5}$$

where F_a , F_θ , F_r , and M_η are the stress and moment resultants in the shell which are functions of the deformations of the shell and where l_1 , l_2 , l_3 , and l_4 contain the contribution of the applied external stress and moment resultants at the shell boundary.

Potential energy of the pressurizing gas.- The equations developed up to this point in the analysis are applicable to rings of both open and closed cross section. It is also useful to extend the analysis to incorporate a pressurized ring of arbitrary closed cross section. For such a cross section, the change in the potential energy of the pressurizing gas must be accounted for in determining the stiffness of the ring. The change in the potential energy of the pressurizing gas has been derived in reference 5 by utilizing the following assumptions:

(1) The pressurized ring is considered to be a membrane in that the local bending stiffness of the walls are neglected.

(2) The internal pressure in the ring is constant during deformation.

In the notation of the present paper and consistent with these assumptions, the potential energy of the pressurizing gas of the ring in terms of the shell variables and ring transverse shear strains is

$$U_p = -p\bar{A}_r \int_0^{2\pi} \left\{ -\frac{\gamma_{yz}^2}{2} - \frac{\gamma_{yx}^2}{2} + \frac{C' - A}{r_c} + \frac{1}{2r_c^2} \left[C^2 + (A')^2 + (B')^2 + 2CA' \right] \right\} r_c d\theta \tag{6}$$

where A , B , and C are given by equations (3).

Shell Boundary Conditions

Derivation of general boundary conditions.- The equilibrium equations of the ring and subsequently the boundary conditions of the shell can be obtained by variation of the total potential energy of the ring. From equations (4), (5), and (6), the result is

$$\begin{aligned}
 \delta U_R + \delta U_P + \delta U_f = & \int_0^{2\pi} \left\{ \left(N - \frac{M_x}{r_c} \right) \delta e_o + \left(M_x - \frac{EI_x}{r_c} e_o \right) \delta \eta_1 + \left(M_z - \frac{EI_{xz}}{r_c} e_o \right) \delta \eta_2 + EI_x e_o \delta \eta_3 \right. \\
 & + EI_z e_o \delta \eta_4 + EI_{xz} e_o \delta \eta_5 + \frac{E\Gamma}{r_s r_c} (b_{20} \phi'' + b_{21} u'') (b_{20} \delta \phi'' + b_{21} \delta u'') \\
 & + A_r G (\gamma_{yz} \delta \gamma_{yz} + \gamma_{yx} \delta \gamma_{yx}) + \frac{GJ}{r_c} r_s \left[(b_{20} \phi' + b_{21} u') (b_{20} \delta \phi' \right. \\
 & \left. + b_{21} \delta u') \right] \left. \right\} r_c d\theta - p \bar{A}_r \int_0^{2\pi} \left[\left(-\gamma_{yz} + \frac{b_{15}}{r_c^2} \xi \right) \delta \gamma_{yz} + \left(-\gamma_{yx} + \frac{b_{17}}{r_c^2} \xi \right) \delta \gamma_{yx} \right. \\
 & + \frac{b_3}{r_c^2} \xi \delta v + \frac{1}{r_c} \left(1 - \frac{p_1}{r_c} \xi \right) \delta w + \frac{1}{r_c^2} (B'' - b_{10} \xi') \delta u + \frac{1}{r_c} \left(b_1 + \frac{b_2}{r_c} B'' \right. \\
 & \left. - \frac{p_2}{r_c} \xi' \right) \delta \phi \left. \right] r_c d\theta + \int_0^{2\pi} \left[(rF_a - l_1) \delta u + (rF_\theta - l_2) \delta v + (rF_r - l_3) \delta w \right. \\
 & \left. + (rM_\eta - l_4) \delta \phi \right] d\theta \tag{7}
 \end{aligned}$$

corresponding to arbitrary variations of the shell middle-surface displacements and rotation δu , δv , δw , and $\delta \phi$ and arbitrary variations of the ring shear strains $\delta \gamma_{yx}$ and $\delta \gamma_{yz}$. Equation (7) can be rearranged to isolate the generalized forces in the ring which correspond to the generalized displacements u , v , w , ϕ , γ_{yx} , and γ_{yz} . For equilibrium of the ring, equation (7) vanishes for arbitrary virtual displacements, leading to the following equations which must be satisfied:

$$rF_a \pm r_c Q_1 = l_1 \quad (\text{Or } \delta u = 0) \tag{8a}$$

$$rF_\theta \pm r_c Q_2 = l_2 \quad (\text{Or } \delta v = 0) \tag{8b}$$

$$rF_r \pm r_c Q_3 = l_3 \quad (\text{Or } \delta w = 0) \tag{8c}$$

$$rM_\eta \pm r_c Q_4 = l_4 \quad (\text{Or } \delta \phi = 0) \tag{8d}$$

$$Q_5 = 0 \quad (\text{Or } \delta\gamma_{yx} = 0) \quad (8e)$$

$$Q_6 = 0 \quad (\text{Or } \delta\gamma_{yz} = 0) \quad (8f)$$

where Q_1, Q_2, \dots, Q_6 are listed in appendix D. The choice of the plus or minus signs in equations (8) depends on the origin chosen for the shell coordinates. Equations (8a) to (8d) become the general nonlinear boundary conditions of an arbitrary shell of revolution with an attached ring provided equations (8e) and (8f) are used to eliminate the shear strains γ_{yz} and γ_{yx} . Before carrying out this elimination, however, it is appropriate to express equations (8) in a linearized form.

Linearization of the boundary conditions.- Many practical problems of aerospace shell structures involve small asymmetric deviations away from an axisymmetric state of equilibrium. The ring boundary conditions will be written herein in a form applicable to such problems wherein the asymmetric linear deviation may be due to either bifurcation buckling or the application of an asymmetric load. To this end, the stress and moment resultants (F_a , F_θ , F_r , and M_η); displacements, rotation, and shear strains (u , v , w , ϕ , γ_{yx} , and γ_{yz}); and external loads (l_i) are expressed as the sum of two parts; for example,

$$F_a = F_{a,A} + F_{a,B} \quad (9)$$

where the term with subscript A describes an initial axisymmetric equilibrium state and the term with subscript B is an incremental change in the term away from this equilibrium state.

The linearized set of boundary conditions for the incremental equilibrium state (subscript B) is obtained by substituting equations typified by equation (9) into equations (8), subtracting out the initial state (subscript A), and neglecting terms nonlinear with respect to subscript B variables. The hoop force, moments, and strain in the ring for the initial and incremental states were also obtained and are listed in appendix D. The linearized subscript B variables are then taken to vary harmonically in the circumferential direction in the following manner. The quantities u_B , w_B , ϕ_B , $F_{a,B}$, $F_{r,B}$, $M_{\eta,B}$, $l_{1,B}$, $l_{3,B}$, and $l_{4,B}$ vary in a cosine distribution; for example,

$$u_B = u_B^{(n)} \cos n\theta \quad (10)$$

and v_B , $\gamma_{yz,B}$, $\gamma_{yx,B}$, $F_{\theta,B}$, and $l_{2,B}$ vary sinusoidally; for example,

$$v_B = v_B^{(n)} \sin n\theta \quad (11)$$

The boundary-condition equations for the incremental state can then be put in matrix form as

$$\begin{array}{c}
 \left. \begin{array}{c}
 F_{a,B}^{(n)} \\
 F_{\theta,B}^{(n)} \\
 F_{r,B}^{(n)} \\
 M_{\eta,B}^{(n)} \\
 0 \\
 0
 \end{array} \right\} \pm r_c
 \end{array}
 \left[\begin{array}{cccccc}
 \bar{G}_{11,A} + \bar{G}_{11,B} & \bar{G}_{12,A} + \bar{G}_{12,B} & \bar{G}_{13,A} + \bar{G}_{13,B} & \bar{G}_{14,A} + \bar{G}_{14,B} & \bar{G}_{15,A} + \bar{G}_{15,B} & \bar{G}_{16,A} + \bar{G}_{16,B} \\
 \text{---} & \bar{G}_{22,A} + \bar{G}_{22,B} & \bar{G}_{23,A} + \bar{G}_{23,B} & \bar{G}_{24,A} + \bar{G}_{24,B} & \bar{G}_{25,A} + \bar{G}_{25,B} & \bar{G}_{26,A} + \bar{G}_{26,B} \\
 \text{---} & \text{---} & \bar{G}_{33,A} + \bar{G}_{33,B} & \bar{G}_{34,A} + \bar{G}_{34,B} & \bar{G}_{35,A} + \bar{G}_{35,B} & \bar{G}_{36,A} + \bar{G}_{36,B} \\
 \text{---} & \text{---} & \text{---} & \bar{G}_{44,A} + \bar{G}_{44,B} & \bar{G}_{45,A} + \bar{G}_{45,B} & \bar{G}_{46,A} + \bar{G}_{46,B} \\
 \text{---} & \text{---} & \text{---} & \text{---} & \bar{G}_{55,A} + \bar{G}_{55,B} & \bar{G}_{56,A} + \bar{G}_{56,B} \\
 \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \bar{G}_{66,A} + \bar{G}_{66,B}
 \end{array} \right]
 \begin{array}{c}
 u_B^{(n)} \\
 v_B^{(n)} \\
 w_B^{(n)} \\
 \phi_B^{(n)} \\
 \gamma_{yz,B}^{(n)} \\
 \gamma_{yx,B}^{(n)}
 \end{array}
 =
 \begin{array}{c}
 \left. \begin{array}{c}
 \lambda_{1,B}^{(n)} \\
 \lambda_{2,B}^{(n)} \\
 \lambda_{3,B}^{(n)} \\
 \lambda_{4,B}^{(n)} \\
 0 \\
 0
 \end{array} \right\}
 \end{array}
 \quad (12)$$

where the $\bar{G}_{ij,A}$ and $\bar{G}_{ij,B}$ are given in equations (E1) and (E2), respectively, in appendix E. The subscripts A and B refer to terms arising from the initial and incremental stress states, respectively.

For special conditions, these general coefficients simplify. For example, for a ring which has no internal pressure, has no restraint of warping, has the shear center and centroid coincident, and is attached at the centroid to the middle surface of the shell, equations (12) reduce to

Equations (14) are convenient for use in a subsequent section to compare results from the present theory with well-known results in the literature for in-plane and out-of-plane ring buckling.

Final form of linearized boundary conditions.- To obtain the stiffness coefficients for the ring in the form of equations (1), the shear strains $\gamma_{yz,B}^{(n)}$ and $\gamma_{yx,B}^{(n)}$ must be eliminated from equations (12). The last two equations of equations (12) give

$$\left. \begin{aligned} \gamma_{yz,B}^{(n)} &= \frac{\tilde{G}_{56}(\tilde{G}_{16}u_B^{(n)} + \tilde{G}_{26}v_B^{(n)} + \tilde{G}_{36}w_B^{(n)} + \tilde{G}_{46}\phi_B^{(n)}) - \tilde{G}_{66}(\tilde{G}_{15}u_B^{(n)} + \tilde{G}_{25}v_B^{(n)} + \tilde{G}_{35}w_B^{(n)} + \tilde{G}_{45}\phi_B^{(n)})}{\tilde{G}_{55}\tilde{G}_{66} - \tilde{G}_{56}^2} \\ \gamma_{yx,B}^{(n)} &= \frac{\tilde{G}_{56}(\tilde{G}_{15}u_B^{(n)} + \tilde{G}_{25}v_B^{(n)} + \tilde{G}_{35}w_B^{(n)} + \tilde{G}_{45}\phi_B^{(n)}) - \tilde{G}_{55}(\tilde{G}_{16}u_B^{(n)} + \tilde{G}_{26}v_B^{(n)} + \tilde{G}_{36}w_B^{(n)} + \tilde{G}_{46}\phi_B^{(n)})}{\tilde{G}_{55}\tilde{G}_{66} - \tilde{G}_{56}^2} \end{aligned} \right\} \quad (15)$$

where

$$\tilde{G}_{ij} = \bar{G}_{ij,A} + \bar{G}_{ij,B} \quad (16)$$

Substitution of equations (15) into the first four equations of equations (12) leads to the desired stiffness coefficients in the form of equations (1) where

$$\begin{aligned}
 G_{11} &= r_c \left(\tilde{G}_{11} + \frac{-\tilde{G}_{15}^2 \tilde{G}_{66} + 2\tilde{G}_{15} \tilde{G}_{16} \tilde{G}_{56} - \tilde{G}_{16}^2 \tilde{G}_{55}}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right) \\
 G_{12} &= r_c \left[\tilde{G}_{12} + \frac{\tilde{G}_{15}(\tilde{G}_{56} \tilde{G}_{26} - \tilde{G}_{66} \tilde{G}_{25}) + \tilde{G}_{16}(\tilde{G}_{56} \tilde{G}_{25} - \tilde{G}_{55} \tilde{G}_{26})}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right] \\
 G_{13} &= r_c \left[\tilde{G}_{13} + \frac{\tilde{G}_{15}(\tilde{G}_{56} \tilde{G}_{36} - \tilde{G}_{66} \tilde{G}_{35}) + \tilde{G}_{16}(\tilde{G}_{56} \tilde{G}_{35} - \tilde{G}_{55} \tilde{G}_{36})}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right] \\
 G_{14} &= r_c \left[\tilde{G}_{14} + \frac{\tilde{G}_{15}(\tilde{G}_{56} \tilde{G}_{46} - \tilde{G}_{66} \tilde{G}_{45}) + \tilde{G}_{16}(\tilde{G}_{56} \tilde{G}_{45} - \tilde{G}_{55} \tilde{G}_{46})}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right] \\
 G_{22} &= r_c \left(\tilde{G}_{22} + \frac{-\tilde{G}_{25}^2 \tilde{G}_{66} + 2\tilde{G}_{25} \tilde{G}_{26} \tilde{G}_{56} - \tilde{G}_{26}^2 \tilde{G}_{55}}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right) \\
 G_{23} &= r_c \left[\tilde{G}_{23} + \frac{\tilde{G}_{25}(\tilde{G}_{56} \tilde{G}_{36} - \tilde{G}_{66} \tilde{G}_{35}) + \tilde{G}_{26}(\tilde{G}_{56} \tilde{G}_{35} - \tilde{G}_{55} \tilde{G}_{36})}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right] \\
 G_{24} &= r_c \left[\tilde{G}_{24} + \frac{\tilde{G}_{25}(\tilde{G}_{56} \tilde{G}_{46} - \tilde{G}_{66} \tilde{G}_{45}) + \tilde{G}_{26}(\tilde{G}_{56} \tilde{G}_{45} - \tilde{G}_{55} \tilde{G}_{46})}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right] \\
 G_{33} &= r_c \left(\tilde{G}_{33} + \frac{-\tilde{G}_{35}^2 \tilde{G}_{66} + 2\tilde{G}_{35} \tilde{G}_{36} \tilde{G}_{56} - \tilde{G}_{36}^2 \tilde{G}_{55}}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right) \\
 G_{34} &= r_c \left[\tilde{G}_{34} + \frac{\tilde{G}_{35}(\tilde{G}_{56} \tilde{G}_{46} - \tilde{G}_{66} \tilde{G}_{45}) + \tilde{G}_{36}(\tilde{G}_{56} \tilde{G}_{45} - \tilde{G}_{55} \tilde{G}_{46})}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right] \\
 G_{44} &= r_c \left(\tilde{G}_{44} + \frac{-\tilde{G}_{45}^2 \tilde{G}_{66} + 2\tilde{G}_{45} \tilde{G}_{46} \tilde{G}_{56} - \tilde{G}_{46}^2 \tilde{G}_{55}}{\tilde{G}_{55} \tilde{G}_{66} - \tilde{G}_{56}^2} \right)
 \end{aligned} \tag{17}$$

If transverse shear strains in the ring can be neglected, equations (17) still define the stiffness coefficients of equations (1) where now

$$\tilde{G}_{15} = \tilde{G}_{16} = \tilde{G}_{25} = \tilde{G}_{26} = \tilde{G}_{35} = \tilde{G}_{36} = \tilde{G}_{45} = \tilde{G}_{46} = 0$$

The form of equations (1) is appropriate for computer programs if the shell behavior is defined in cylindrical coordinates. The transformation required for shells defined in the intrinsic shell coordinates is given in appendix F.

DISCUSSION OF RESULTS

Boundary-Condition Equations

Equations (1) are the general boundary conditions for an arbitrary shell of revolution with an elastic ring of general cross section rigidly attached at the shell boundary. These equations as developed in this paper can be used for the stress analysis of symmetrically or asymmetrically loaded shells with a symmetrically prestressed ring attached at the shell boundary. The terms l_i represent the external loads, if any, applied at the shell boundary.

When the equations are applied to stability problems, the external loading on the shell must be symmetric so that the prebuckling stresses and deformations of both the shell and the ring are also symmetric. (This restriction, of course, does not imply that the buckling deformations must be symmetric.) The prestress terms in the stiffness matrix are now obtained from the axisymmetric prebuckling solution. Also, for buckling problems, the l_i are always zero if the external loads on the ring are zero.

An exception to the previous discussion must be noted for the special case of the pressurized ring. For the pressurized ring, the boundary conditions given by equations (1) are valid only for axisymmetric stress analysis and symmetric or asymmetric buckling analysis. Equations (1) are not valid for asymmetric stress analysis of a pressurized ring because of the way in which the internal pressure terms were handled in the perturbation procedure. (When nonhomogeneous terms appear on the right-hand sides of eqs. (8), eqs. (10) and (11) are not valid solutions for $n > 0$.)

Buckling Solutions for a Ring

To check the validity of the ring stiffness coefficients developed in this paper, selected ring buckling calculations were carried out for a ring of doubly symmetric cross section subjected to a constant radial line loading \hat{q} . Equations (14) are in a convenient form to consider both in-plane and out-of-plane buckling. The axisymmetric prestress deformations are obtained from equations (14) by setting $n = 0$, letting $\lambda_{3,B}^{(n)} = -r_c \hat{q}$, and replacing the subscript B with A so that

$$\left. \begin{aligned}
 w_A &= -\frac{\hat{q}r_c^2}{EA_R + \frac{EI_x}{r_c^2}} \\
 \gamma_{yz,A} &= 0 \\
 \phi_A &= 0 \\
 \gamma_{yx,A} &= 0
 \end{aligned} \right\} \quad (18)$$

The other two prestress deformations v_A and u_A vanish from axisymmetry and lack of rigid body motion, respectively. The prestress strain, hoop force, and moments may be determined by substitution of equations (18) and (3) into equations (D4) which yields

$$\left. \begin{aligned}
 e_{o,A} &= -\frac{\hat{q}r_c}{EA_R + \frac{EI_x}{r_c^2}} \\
 N_A &= -\hat{q}r_c \\
 M_{x,A} &= -\frac{EI_x \hat{q}}{EA_R + \frac{EI_x}{r_c^2}} \\
 M_{z,A} &= 0
 \end{aligned} \right\} \quad (19)$$

Since in the formulation of the buckling problem the incremental loads are zero, the determinant of the coefficients of the displacements of equations (14) must equal zero. Because $M_{z,A}$ is equal to zero, in-plane buckling and out-of-plane buckling are uncoupled.

In-plane buckling. - The in-plane buckling solutions are obtained by setting the determinant of the coefficients of the in-plane variables of equations (14) equal to zero. A first-order approximation to the critical load is obtained by retaining only the linear terms in \hat{q} in the expansion for the determinant. For the lowest buckling mode, $n = 2$, this expansion yields

$$\frac{\hat{q}r_c^3}{EI_x} = \frac{4}{1 + \frac{2}{S} \frac{T+3}{T+1}} = \frac{4}{1 + \frac{2}{S} + \frac{4}{S} \frac{1}{T+1}} \quad (20)$$

where

$$\left. \begin{aligned} T &= \frac{EA_r r_c^2}{EI_x} \\ S &= \frac{GA_r r_c^2}{2EI_x} \end{aligned} \right\} \quad (21)$$

The critical load as determined by equation (20) agrees with the result given by Ratzersdorfer (ref. 6) when $S \rightarrow \infty$.

Out-of-plane buckling. - The out-of-plane buckling solutions are obtained by setting the determinant of the coefficients of the out-of-plane variables of equations (14) equal to zero. Again a first-order approximation to the critical load is obtained by retaining only the linear terms in \hat{q} in the expansion. For the lowest buckling mode, $n = 2$, this expansion yields

$$\frac{\hat{q} r_c^3}{EI_z} = \frac{9}{4 + \frac{1}{\psi} + \frac{8}{fS} - \frac{1}{T+1} \left(4 + \frac{1}{\psi} + \frac{8}{fS} \right) + \frac{3}{T+1} \left[\frac{1}{2f\psi} (-4f + 6g) + \frac{3}{2f^2 S} \right]} \quad (22)$$

where

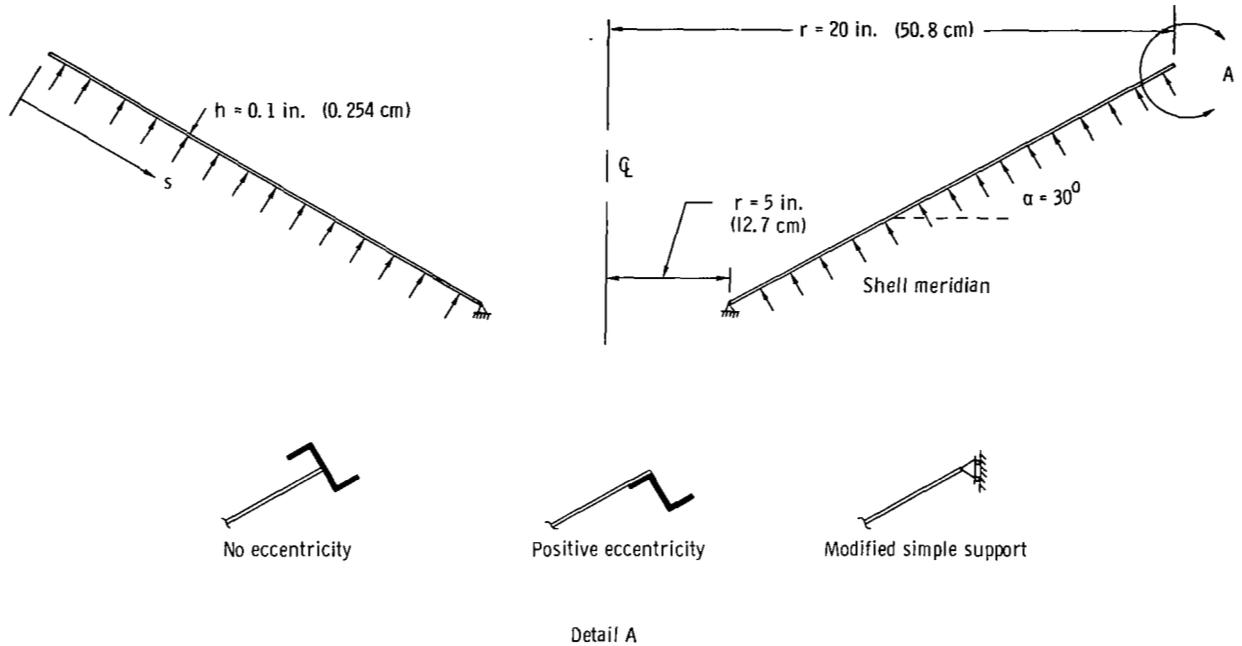
$$\left. \begin{aligned} \psi &= \frac{GJ}{EI_z} \\ f &= \frac{EI_x}{EI_z} \\ g &= \frac{EI_x + EI_z}{EI_z} \end{aligned} \right\} \quad (23)$$

When $T \rightarrow \infty$ and $S \rightarrow \infty$, equation (22) reduces to the classical solution contained in reference 7.

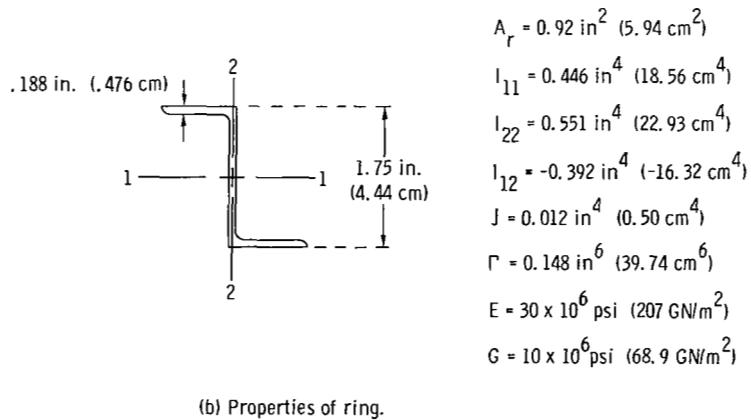
Application to the Stress Analysis of a Conical Shell

To illustrate the application of the ring theory to a shell problem, the boundary-condition matrices were used to solve a representative shell stress-analysis problem. The problem chosen is a conical shell subjected to a normal pressure loading q which is constant along the meridian and varies harmonically around the circumference of the shell (fig. 2(a)). Thus, $q = q_n \cos n\theta$ where q_n is a constant. This loading is appropriate because a general asymmetric load on a shell of revolution can be expanded into a series of such components.

The shell is simply supported at the small end (that is, all displacements and the meridional bending-moment resultant vanish) and contains a Z-section ring rigidly attached at the large end. The properties of the ring are shown in figure 2(b).



(a) Geometry of conical shell with detail of various ring attachment positions.



(b) Properties of ring.

Figure 2.- Geometry of conical shell and edge-stiffening ring.

Results were obtained from the computer program of reference 8 for the distribution of the nondimensional shell variables,

$$m_\eta = \frac{M_\eta(n)}{q_n h^2}$$

$$t_\eta = \frac{N_\eta(n)}{q_n r_0}$$

$$w_\eta = \frac{Eh}{q_n r_0^2} \tilde{w}(n)$$

where r_0 is the radius of the middle surface of the conical shell at the large end. Poisson's ratio was taken to be 0.3 in the calculations and the shell and ring were made of the same material. The amplitudes of meridional moment resultant, meridional stress resultant, and normal displacement as functions of distance from the stiffened edge (up to 0.6 of s_{\max}) are shown in figures 3, 4, and 5 for the Fourier indices $n = 3$ and 8. The effects of eccentricity are shown in the figures for the ring centroid attached to the shell ($z_0 = 0$) and for a leg of the ring attached to the outside of the shell ($z_0 = 0.925$ inch (2.35 cm)).

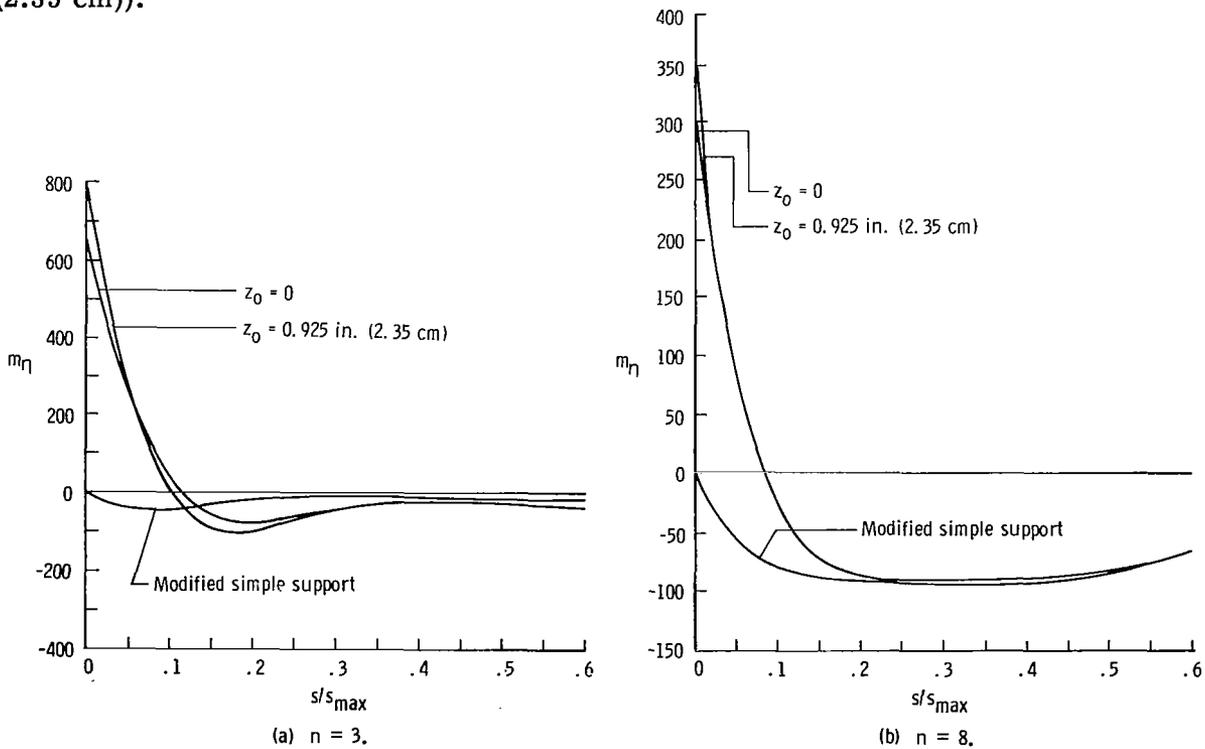


Figure 3.- Nondimensional meridional moment resultant m_η as a function of distance from stiffened edge of conical shell which is subjected to external pressure.

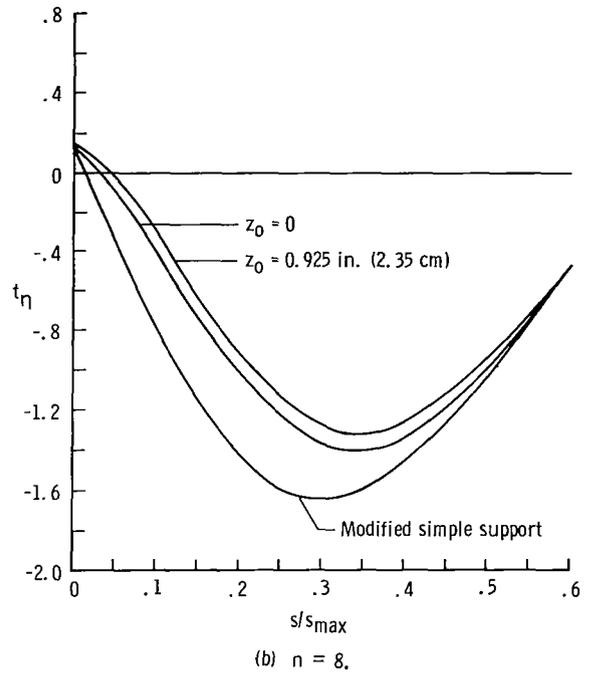
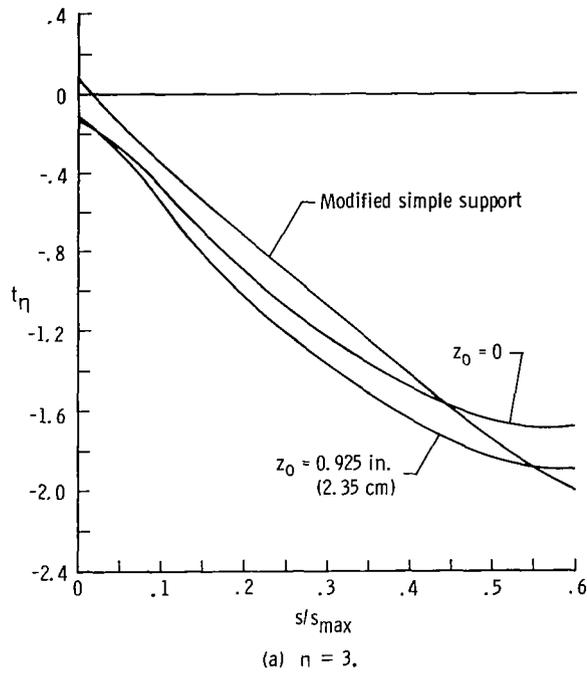


Figure 4.- Nondimensional meridional stress resultant t_η as a function of distance from stiffened edge of conical shell which is subjected to external pressure.

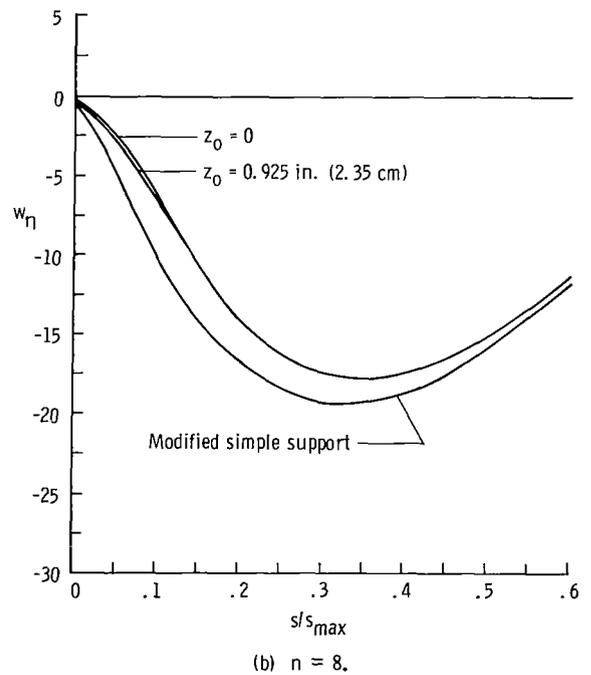
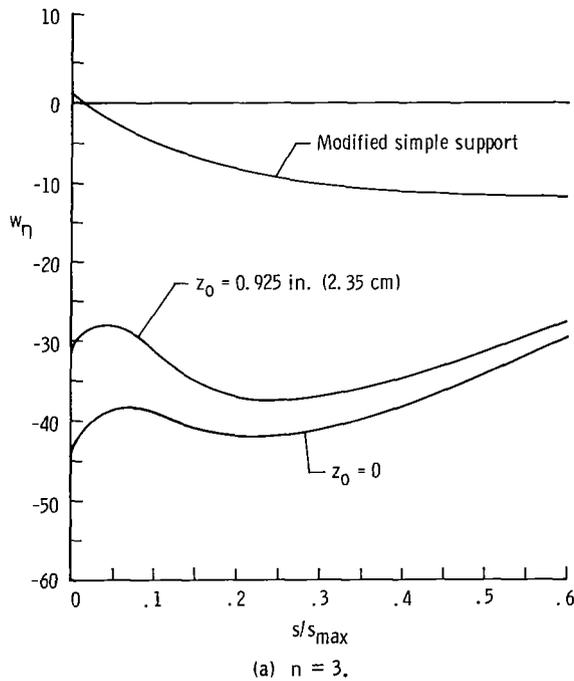


Figure 5.- Nondimensional normal displacement w_η as a function of distance from stiffened edge of conical shell which is subjected to external pressure.

Results are also shown in the figures for the shell with the same boundary conditions at the small end and with boundary conditions denoted as "modified simple support" at the large end. The modified-simple-support boundary conditions are (see fig. 2(a)) as follows: Radial motion is restrained, movement in the direction of the axis of the cone is unrestrained, meridional moment vanishes, and circumferential displacement vanishes. These boundary conditions could be used to approximate the contribution of a stiff ring to shell behavior.

Results were also obtained for the same ring-stiffened shell problems by using the ring boundary conditions contained in reference 1. The calculations based on the ring theory of reference 1 agree with the present calculations to within 5 percent and consequently are not shown in the figures. Further comparisons of results obtained with the two theories are given in reference 5.

Plots for $n = 3$ in figures 3, 4, and 5 show that taking into account the eccentricity of the ring can cause appreciable changes in the behavior of the shell. Ring eccentricity, however, was not important for $n = 8$. The results also show that the modified-simple-support boundary conditions for the shell are poor approximations to the behavior of the ring-stiffened shell. The approximation for stress resultants appears to be better than that for displacements and moments.

CONCLUDING REMARKS

The contributions of ring stiffness to the boundary conditions for asymmetric behavior of a prestressed shell of revolution have been derived for an elastic ring of arbitrary cross section rigidly attached to the shell boundary. The ring behavior includes the effects of shear deformation, restraint of warping, ring torsion, out-of-plane bending, internal pressure, eccentricity, and axisymmetric prestress. The results for the ring stiffness matrix are presented in a form that is well suited for inclusion in a computer program for analysis of shells of revolution. The ring stiffness effects are applied to the sample case of a conical shell stiffened at one end by a ring and subjected to an asymmetric load. The results show that it is important to include properly the ring effect and ring eccentricity in order to determine accurately the stresses in the shell.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., December 9, 1969.

APPENDIX A

CONVERSION OF U.S. CUSTOMARY UNITS TO SI UNITS

The International System of Units (SI) was adopted by the Eleventh General Conference on Weights and Measures, Paris, 1960 (ref. 3). Factors for converting the U.S. Customary Units used herein to the International System of Units are given in the following table:

Physical quantity	U.S. Customary Unit	Conversion factor (*)	SI Unit (**)
Area	in ²	6.452×10^{-4}	meters ² (m ²)
Length	in.	2.54×10^{-2}	meters (m)
Moment of inertia . . .	in ⁴	4.162×10^{-7}	meters ⁴ (m ⁴)
Stress	psi = lbf/in ²	6.895×10^3	newtons/meter ² (N/m ²)
Warping constant . . .	in ⁶	2.685×10^{-10}	meters ⁶ (m ⁶)

*Multiply value given in U.S. Customary Unit by conversion factor to obtain equivalent value in SI Unit.

**Prefixes to indicate multiples of SI Units are as follows:

Prefix	Multiple
giga (G)	10 ⁹
centi (c)	10 ⁻²

APPENDIX B

DERIVATION OF THE KINEMATIC RELATIONS FOR A CIRCULAR RING ATTACHED TO A SHELL OF REVOLUTION

The kinematic relations are derived for a circular ring of arbitrary cross section undergoing shear deformations and rigidly attached to the surface of a shell of revolution. These relations are then expressed in terms of the middle-surface shell displacements and rotation in a cylindrical coordinate system by requiring compatibility of the ring and shell at the point of attachment. (A more detailed analysis may be found in ref. 5.)

Displacements of a Point in the Cross Section of a Circular Ring

The coordinate system to be used for the ring (fig. B1) is as follows: a coordinate x normal to the plane of the ring, a coordinate y along the shear-center axis of the ring so that $y = r_s \theta$, and a coordinate z measured radially inward from the shear center. The subscript s refers to the shear center. The displacement vector at any point in the ring is made up of the displacement

components u_a , v_a , and w_a in the directions x , y , and z , respectively. The rotation β is assumed to be positive clockwise about the y -axis through the shear centers. (The positive rotation vector is tangent to the ring circumference in the plane of the ring and in the direction of increasing θ .)

In defining β , the thickness-compression strains e_x and e_z are assumed to be zero so that the projection of the deformed ring cross section into the xz -plane remains invariant during deformation. Note that this assumption does not preclude a warping displacement normal to the xz -plane. Consistent with this assumption, the shear strain e_{xz} is taken to be zero throughout this analysis.

The displacements u_a , v_a , and w_a for moderately small rotations are, to first order,

$$u_a(z, \theta) = u_s(\theta) + z\beta(\theta) \quad (B1a)$$

$$v_a(x, z, \theta) = \frac{r_s - z}{r_s} v_s(\theta) - \frac{z}{r_s} w_s'(\theta) - \frac{x}{r_s} u_s'(\theta) - z\gamma_{yz}(\theta) - x\gamma_{yx}(\theta) \quad (B1b)$$

$$w_a(x, \theta) = w_s(\theta) - x\beta(\theta) \quad (B1c)$$

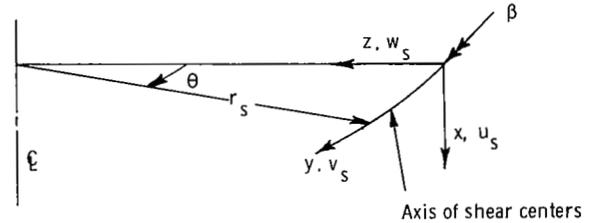


Figure B1.- Ring coordinate system.

APPENDIX B – Continued

where the last two terms for v_a in equation (B1b) are the effects of transverse shear deformations of the ring. (See ref. 5.) Note that equations (B1) do not include a contribution from warping of the cross section. This contribution is considered separately in appendix C.

For a thin shell, the displacements in a cylindrical coordinate system (fig. B2) are

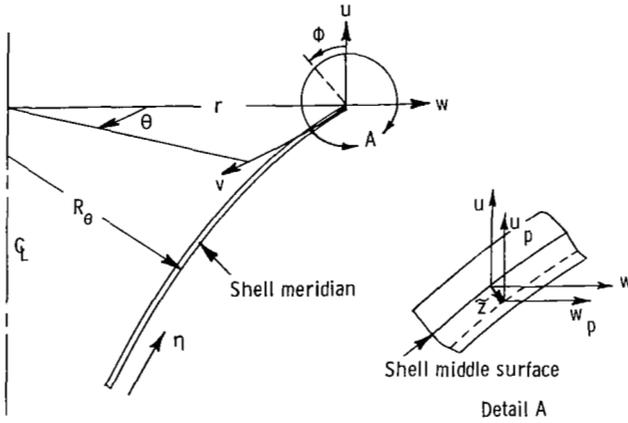


Figure B2.- Cylindrical coordinate system for arbitrary shell of revolution.

$$\left. \begin{aligned} u_p(\theta, \eta, \tilde{z}) &= u(\theta, \eta) + a_5 \tilde{z} \phi(\theta, \eta) \\ v_p(\theta, \eta, \tilde{z}) &= v(\theta, \eta) + \tilde{z} [a_1 w'(\theta, \eta) \\ &\quad + a_2 u'(\theta, \eta) + a_3 v(\theta, \eta)] \\ w_p(\theta, \eta, \tilde{z}) &= w(\theta, \eta) + a_4 \tilde{z} \phi(\theta, \eta) \end{aligned} \right\} \quad (B2)$$

where

$$\left. \begin{aligned} a_1 &= -\frac{1}{R_\theta} & a_3 &= -a_1 \\ a_2 &= \frac{dr/d\eta}{r} & a_4 &= ra_2 \\ & & a_5 &= -ra_1 \end{aligned} \right\} \quad (B3)$$

and r is the radius of the shell middle surface and R_θ is the radius of curvature which generates the shell circumference. Here, u_p , v_p , and w_p are shell displacements in the axial, circumferential, and radial directions, respectively, at a distance \tilde{z} from the shell middle surface, and u , v , and w are the corresponding displacements at the middle surface.

Compatibility of Ring and Shell Displacements at the Shell Boundary

The ring displacements given by equations (B1) are now expressed in terms of the shell variables given by equations (B2) by requiring the displacements and rotation of the ring and shell to be the same at the point of attachment. (See fig. B3.) The line of attachment of the ring to the shell is at $\eta = \text{Constant}$. Therefore, at the point of attachment, the following conditions must be satisfied:

$$\left. \begin{aligned} u_a(\theta, z) &= -u_p(\theta, \eta, \tilde{z}) \\ v_a(\theta, x, z) &= v_p(\theta, \eta, \tilde{z}) \\ w_a(\theta, x) &= -w_p(\theta, \eta, \tilde{z}) \\ \beta(\theta) &= \phi(\theta) \end{aligned} \right\} \quad (\tilde{z} = z_{sh}, \quad x = \bar{x}, \quad z = \bar{z}) \quad (B4)$$

APPENDIX B - Continued

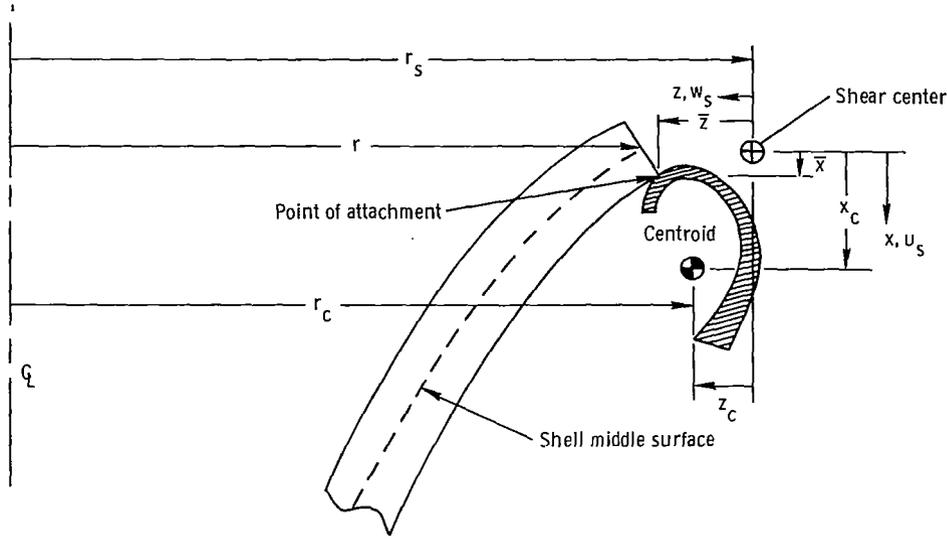


Figure B3.- Geometry of attachment of ring to shell. (Ring is attached to shell at $\bar{z} = z_{sh} = h/2$.)

Here z_{sh} is the distance measured normal from the shell middle surface to the point of attachment, and \bar{x} and \bar{z} are the axial and radial distances from the shear center of the ring to the point of attachment. By means of equations (B2) and (B4), the displacements at any point in the ring cross section u_a , v_a , and w_a can be expressed in terms of the shell variables and ring shear strains. By grouping in powers of $x - x_c$ and $z - z_c$, the result is as given by equations (2) and (3) in the body of the paper where

$$b_1 = \bar{x} - x_c - a_4 z_{sh} \quad (B5a)$$

$$b_2 = \bar{z} - z_c + a_5 z_{sh} \quad (B5b)$$

$$b_3 = \frac{r_c}{r_s - \bar{z}} (1 + a_3 z_{sh}) \quad (B5c)$$

$$b_4 = -\frac{b_3}{r_c} \quad (B5d)$$

$$b_5 = \frac{r_c}{r_s - \bar{z}} \left(a_1 z_{sh} - \frac{\bar{z}}{r_s} \right) + \frac{z_c}{r_s} \quad (B5e)$$

$$b_6 = -\frac{1}{r_s - \bar{z}} \left(a_1 z_{sh} - \frac{\bar{z}}{r_s} \right) + \frac{1}{r_s} \quad (B5f)$$

$$b_7 = \frac{1}{r_s} (\bar{x} - a_4 z_{sh}) \left(\frac{r_c \bar{z}}{r_s - \bar{z}} - z_c \right) + \frac{1}{r_s} (\bar{z} + a_5 z_{sh}) \left(\frac{-r_c \bar{x}}{r_s - \bar{z}} + x_c \right) \quad (B5g)$$

$$b_8 = -\frac{1}{r_s} (\bar{x} - a_4 z_{sh}) \left(\frac{\bar{z}}{r_s - \bar{z}} + 1 \right) + \frac{\bar{x}}{r_s (r_s - \bar{z})} (\bar{z} + a_5 z_{sh}) \quad (B5h)$$

$$b_9 = \frac{1}{r_s} (\bar{z} + a_5 z_{sh}) \quad (B5i)$$

APPENDIX B - Concluded

$$b_{10} = \frac{r_c}{r_s - \bar{z}} \left(a_2 z_{sh} - \frac{\bar{x}}{r_s} \right) + \frac{x_c}{r_s} \quad (B5j)$$

$$b_{11} = \frac{1}{r_s - \bar{z}} \left(a_2 z_{sh} - \frac{\bar{x}}{r_s} \right) \quad (B5k)$$

$$b_{12} = \frac{1}{r_s} \quad (B5l)$$

$$b_{13} = 0 \quad (B5m)$$

$$b_{14} = 0 \quad (B5n)$$

$$b_{15} = \frac{r_c}{r_s - \bar{z}} \bar{z} - z_c \quad (B5o)$$

$$b_{16} = -\frac{\bar{z}}{r_s - \bar{z}} - 1 \quad (B5p)$$

$$b_{17} = \frac{r_c}{r_s - \bar{z}} \bar{x} - x_c \quad (B5q)$$

$$b_{18} = -1 \quad (B5r)$$

$$b_{19} = -\frac{\bar{x}}{r_s - \bar{z}} \quad (B5s)$$

In equations (B5), x_c and z_c are the axial and radial distances, respectively, from the shear center to the centroid of the ring, and r_c is the radius to the centroid (fig. B3).

APPENDIX C

STRAIN ENERGY FOR A CIRCULAR RING IN TERMS OF SHELL VARIABLES

Strain Energy of Bending and Extension

The nonlinear strain-displacement relation for the circumferential strain at any point in the ring in terms of the ring displacements (ref. 9) is, in the notation of the present paper,

$$e_y = \frac{1}{r_s - z}(v'_a - w_a) + \frac{1}{2(r_s - z)^2} \left[(u'_a)^2 + (v_a + w'_a)^2 \right] \quad (C1)$$

By substituting equations (2) into equation (C1), the circumferential strain of the ring is expressed in terms of the shell variables and ring shear strains. If higher than second-order terms in z and x are neglected, the result is

$$e_y = e_0 + (z - z_c)\eta_1 + (x - x_c)\eta_2 + (z - z_c)^2\eta_3 + (x - x_c)^2\eta_4 + (x - x_c)(z - z_c)\eta_5 \quad (C2)$$

where

$$e_0 = \frac{\sqrt{\lambda_1}}{r_c} + \frac{\bar{A}}{2r_c^2} \quad (C3a)$$

$$\eta_1 = -\frac{\sqrt{\lambda_2}}{r_c} + \frac{\sqrt{\lambda_1}}{r_c^2} + \frac{\bar{B}}{2r_c^2} + \frac{\bar{A}}{r_c^3} \quad (C3b)$$

$$\eta_2 = -\frac{\sqrt{\lambda_3}}{r_c} + \frac{\bar{C}}{2r_c^2} \quad (C3c)$$

$$\eta_3 = -\frac{\sqrt{\lambda_2}}{r_c^2} + \frac{\sqrt{\lambda_1}}{r_c^3} + \frac{\bar{D}}{2r_c^2} + \frac{\bar{B}}{r_c^3} + \frac{3}{2} \frac{\bar{A}}{r_c^4} \quad (C3d)$$

$$\eta_4 = \frac{\bar{E}}{2r_c^2} \quad (C3e)$$

$$\eta_5 = -\frac{\sqrt{\lambda_3}}{r_c^2} + \frac{\bar{F}}{2r_c^2} + \frac{\bar{C}}{r_c^3} \quad (C3f)$$

APPENDIX C - Continued

and where

$$\left. \begin{aligned}
 \sqrt{\lambda_1} &= C' - A \\
 \sqrt{\lambda_2} &= \frac{D'}{r_s} \\
 \sqrt{\lambda_3} &= \frac{F'}{r_s} - \phi \\
 \bar{A} &= (B')^2 + (C + A')^2 \\
 \bar{B} &= 2B'\phi' - 2\frac{D}{r_s}(C + A') \\
 \bar{C} &= -2(C + A')\left(\frac{F}{r_s} + \phi'\right) \\
 \bar{D} &= (\phi')^2 + \frac{D^2}{r_s^2} \\
 \bar{E} &= \left(\frac{F}{r_s} + \phi'\right)^2 \\
 \bar{F} &= 2\frac{D}{r_s}\left(\frac{F}{r_s} + \phi'\right)
 \end{aligned} \right\} \quad (C4)$$

In equations (C4) the variables A, B, C, D, and F are given by equations (3).

The strain energy for bending and extension of the ring due to the circumferential strain e_y is

$$U_1 = \frac{E}{2} \int_V e_y^2 dV = \frac{E}{2} \int_V e_y^2 dA r [r_c - (z - z_c)] d\theta \quad (C5)$$

where V is the volume of ring material. Equation (C5) is expressed in terms of the shell variables by using equation (C2). Integration over the cross section of the ring yields

$$U_1 = \frac{E}{2} \int_0^{2\pi} \left[A_r e_o^2 + I_x \left(\eta_1^2 + 2\eta_3 e_o - \frac{2}{r_c} e_o \eta_1 \right) + I_z \left(\eta_2^2 + 2e_o \eta_4 \right) + 2I_{xz} \left(e_o \eta_5 + \eta_1 \eta_2 - \frac{e_o \eta_2}{r_c} \right) \right] r_c d\theta \quad (C6)$$

APPENDIX C - Continued

where use has been made of the following standard definitions of moments of inertia and area:

$$\left. \begin{aligned} \int_{A_R} dA_R &= A_R \\ \int_{A_R} (x - x_C) dA_R &= 0 \\ \int_{A_R} (z - z_C) dA_R &= 0 \\ \int_{A_R} (x - x_C)^2 dA_R &= I_Z \\ \int_{A_R} (z - z_C)^2 dA_R &= I_X \\ \int_{A_R} (x - x_C)(z - z_C) dA_R &= I_{XZ} \end{aligned} \right\} \quad (C7)$$

Equation (C6) can be expressed in terms of the ring hoop force and moments. First,

$$\left. \begin{aligned} N &= \int_{A_R} Ee_y dA_R \\ M_X &= \int_{A_R} Ee_y(z - z_C) dA_R \\ M_Z &= \int_{A_R} Ee_y(x - x_C) dA_R \end{aligned} \right\} \quad (C8)$$

where N is the ring hoop force and M_X and M_Z are the ring bending moments about the x - and z -axes, respectively. Then, equations (C8) are integrated by substitution of equation (C2) into equations (C8) and by making use of equations (C7). The result is

$$\left. \begin{aligned} N &= E(A_R e_0 + I_X \eta_3 + I_Z \eta_4 + I_{XZ} \eta_5) \\ M_X &= E(I_X \eta_1 + I_{XZ} \eta_2) \\ M_Z &= E(I_Z \eta_2 + I_{XZ} \eta_1) \end{aligned} \right\} \quad (C9)$$

APPENDIX C – Continued

Finally, equations (C9) can be incorporated into equation (C6) to give the strain energy of the ring resulting from extension and bending as

$$U_1 = \frac{E}{2} \int_0^{2\pi} \left(\frac{2Ne_0}{E} - A_r e_0^2 - \frac{2}{r_c} \frac{e_0}{E} M_x + \frac{M_x}{E} \eta_1 + \frac{M_z}{E} \eta_2 \right) r_c d\theta \quad (C10)$$

The first two terms of the integral in equation (C10) represent the energy associated with stretching of the ring centroid line, the last two terms represent the energy associated with the in-plane and out-of-plane bending, and the third term represents a coupling of the bending and extensional behavior caused by ring curvature.

Strain Energy of Restraint of Warping

A rigorous development for the strain energy of restraint of warping would require a knowledge of the distribution of the warping displacements over the cross section of the ring and would constitute an extremely detailed and complicated analysis. Since restraint of warping is not a primary effect in ring behavior, a first approximation to restraint of warping was felt to be adequate for defining shell boundary conditions. This first approximation of the strain energy of restraint of warping is in the same form as that for a straight beam (ref. 7); namely,

$$U_2 = \frac{E\Gamma}{2r_s r_c} \int_0^{2\pi} (\zeta')^2 r_c d\theta \quad (C11)$$

where ζ is the twist of the ring with respect to the shear center and Γ is the warping constant of straight-beam theory. Values of Γ for cross sections of various shapes can be found in numerous structural handbooks. It might be noted that reference 10 also obtained the expression for strain energy of restraint of warping of a ring by making certain simplifying assumptions on ring behavior.

The twist of the ring (ref. 7) is expressed in the notation of the present paper by

$$\zeta = \frac{1}{r_s} \left(\beta' + \frac{1}{r_s} u_s' \right) \quad (C12)$$

which is expressed in terms of the shell variables by use of equations (2), (3), (B4), and (B5). The result is

$$\zeta = b_{20} \phi' + b_{21} u' \quad (C13)$$

where

$$\left. \begin{aligned} b_{20} &= \frac{1}{r_s} (1 - b_g) \\ b_{21} &= -\frac{1}{r_s^2} \end{aligned} \right\} \quad (C14)$$

Substituting equation (C13) into equation (C11) gives the strain energy of restraint of warping in terms of the shell variables; namely,

$$U_2 = \frac{E\Gamma}{2r_s r_c} \int_0^{2\pi} (b_{20}\phi'' + b_{21}u'')^2 r_c d\theta \quad (C15)$$

Strain Energy of Shear

In this analysis, the strain energy of shear is

$$U_3 = \frac{G}{2} \int_V (e_{yz}^2 + e_{yx}^2) \left[r_c - (z - z_c) \right] dA_r d\theta \quad (C16)$$

where V is the volume of ring material. It is convenient to integrate equation (C16) by separating the total shear strains into two parts – one part due to bending which is a function only of θ and another part due to torsion which is a function of θ , x , and z . The result is

$$\left. \begin{aligned} e_{yz} &= -\gamma_{yz}(\theta) + \bar{\gamma}_{yz}(\theta, x, z) \\ e_{yx} &= -\gamma_{yx}(\theta) + \bar{\gamma}_{yx}(\theta, x, z) \end{aligned} \right\} \quad (C17)$$

where the γ_{yz} and γ_{yx} are the shear terms resulting from bending (the same as those used in eqs. (B1)) and the $\bar{\gamma}_{yz}$ and $\bar{\gamma}_{yx}$ are the remaining shear terms resulting from torsion. By substitution of equations (C17) into equation (C16), the strain energy of shear becomes

$$U_3 = \frac{A_r G}{2} \int_0^{2\pi} (\gamma_{yx}^2 + \gamma_{yz}^2) r_c d\theta + \frac{G}{2} \int_V (-2\gamma_{yz}\bar{\gamma}_{yz} - 2\gamma_{yx}\bar{\gamma}_{yx} + \bar{\gamma}_{yz}^2 + \bar{\gamma}_{yx}^2) dV \quad (C18)$$

where V is the volume of ring material. The first integral on the right-hand side of equation (C18) represents the strain energy due to bending whereas the second integral represents the strain energy due to torsion plus a coupling of torsional and bending shear.

APPENDIX C – Concluded

Clearly, these coupling terms are zero if the torsional strains are zero. Hence, if the coupling effect is neglected, the last integral reduces to the strain energy of pure torsion. As in the case of restraint of warping, a rigorous development of this part of the strain energy would necessitate a detailed study in itself. It is felt that this detailed study would not add materially to the results of the analysis and, consequently, the following simplified expression, similar to that used in straight-beam theory (ref. 7), is used for the strain energy of torsion:

$$U_{\text{torsion}} = \frac{GJ}{2} \frac{r_s}{r_c} \int_0^{2\pi} \xi^2 r_c \, d\theta \quad (\text{C19})$$

so that the total strain energy of shear now becomes

$$U_3 = \frac{A_r G}{2} \int_0^{2\pi} \left(\gamma_{yz}^2 + \gamma_{yx}^2 \right) r_c \, d\theta + \frac{GJ}{2} \frac{r_s}{r_c} \int_0^{2\pi} \left(b_{20}\phi' + b_{21}u' \right)^2 r_c \, d\theta \quad (\text{C20})$$

where J is the torsional constant of straight-beam theory.

In summary, the total strain energy for a ring of arbitrary cross section rigidly attached to a shell of revolution is given by the sum of equations (C10), (C15), and (C20). The result is given by equation (4) in the body of the paper.

APPENDIX D

LIST OF FUNCTIONS

The following is a partial list of functions which are referred to in the main body of the paper and are included for completeness:

$$\begin{aligned}
 Q_1 = & -\frac{1}{r_c^2} \left[-B' + b_{10} \xi \right] \left(N - \frac{M_x}{r_c} \right)' + \frac{b_{10}}{r_c} \left(N - \frac{M_x}{r_c} \right)'' + \frac{p_{10}}{r_c} M_x'' - \frac{1}{r_c^2} \left[-\phi' - \frac{2}{r_c} B' - \frac{b_{10}}{r_s} D + p_{13} \xi \right] \left(M_x - \frac{EI_x e_0}{r_c} \right)' + \frac{b_{12}}{r_c} M_z'' \\
 & - \frac{1}{r_c^2} \left[b_{12} \xi - b_{10} \Lambda \right] \left(M_z - \frac{EI_{xz} e_0}{r_c} \right)' - \frac{1}{r_c^2} \left[\frac{p_{19}}{r_c} \xi - \frac{p_{13}}{r_s} D - \frac{3B'}{r_c^2} - \frac{2\phi'}{r_c} \right] EI_x e_0' + \frac{E\Gamma}{r_s r_c} b_{21} (b_{20} \phi'''' + b_{21} u'''') \\
 & - \frac{GJ}{r_c} r_s b_{21} (b_{20} \phi'' + b_{21} u'') - \frac{\bar{p} \bar{A}_r}{r_c^2} (B'' - b_{10} \xi') + \frac{b_{12}}{r_c^2} EI_z (\Lambda e_0)' + \frac{1}{r_c^2} \left[p_{13} \Lambda + b_{12} \frac{D}{r_s} - 2b_{12} \frac{\xi}{r_c} \right] EI_{xz} e_0' \quad (D1a)
 \end{aligned}$$

$$\begin{aligned}
 Q_2 = & \frac{b_3}{r_c^2} \xi \left(N - \frac{M_x}{r_c} \right) - \frac{b_3}{r_c} \left(N - \frac{M_x}{r_c} \right)' - \frac{p_3}{r_c} M_x' + \frac{1}{r_c^2} \left(p_4 \xi - \frac{b_3}{r_s} D \right) \left(M_x - \frac{EI_x e_0}{r_c} \right) - \frac{b_3}{r_c^2} \Lambda \left(M_z - \frac{EI_{xz} e_0}{r_c} \right) + \frac{1}{r_c^2} \left(\frac{p_{17}}{r_s} \xi - \frac{p_4}{r_s} D \right) EI_{xz} e_0 \\
 & - \frac{p_4}{r_c^2} \Lambda EI_{xz} e_0 - \bar{p} \bar{A}_r \frac{b_3}{r_c^2} \xi \quad (D1b)
 \end{aligned}$$

$$\begin{aligned}
 Q_3 = & \frac{N}{r_c} - \frac{p_1}{r_c^2} \left[\xi \left(N - \frac{M_x}{r_c} \right) \right]' + \frac{b_5}{r_c} \left(N - \frac{M_x}{r_c} \right)'' + \frac{p_7}{r_c} M_x'' - \frac{1}{r_c^2} \left[p_5 \xi - \frac{p_1}{r_s} D \right] \left(M_x - \frac{EI_x e_0}{r_c} \right)' + \frac{p_1}{r_c^2} \left[\Lambda \left(M_z - \frac{EI_{xz} e_0}{r_c} \right) \right]' - \frac{1}{r_c^2} \left[\frac{p_{18}}{r_c} \xi \right. \\
 & \left. - \frac{p_5}{r_s} D \right] EI_x e_0' + \frac{p_5}{r_c^2} EI_{xz} (\Lambda e_0)' - p \frac{\bar{A}_r}{r_c} \left(1 - \frac{p_1}{r_c} \xi' \right) \quad (D1c)
 \end{aligned}$$

APPENDIX D

$$\begin{aligned}
Q_4 = & -\frac{b_1}{r_c} N - \frac{1}{r_c^2} \left[(-b_2 B' + p_2 \xi) \left(N - \frac{M_x}{r_c} \right) \right]' + \frac{b_7}{r_c} \left(N - \frac{M_x}{r_c} \right)'' + \frac{p_6}{r_c} M_x'' - \frac{1}{r_c^2} \left[\left(p_8 B' - b_2 \phi' + p_9 \xi - \frac{p_2}{r_s} D \right) \left(M_x - \frac{EI_x}{r_c} e_0 \right) \right]' \\
& + \frac{b_9}{r_c} M_z'' + \frac{1}{r_c} M_z - \frac{1}{r_c^2} \left[\left(p_{16} \xi - p_2 \Lambda \right) \left(M_z - \frac{EI_{xz}}{r_c} e_0 \right) \right]' - \frac{1}{r_c^2} \left[\left(\frac{p_{20}}{r_c} \xi - \frac{p_9}{r_s} D + p_8 \phi' + \frac{p_{21}}{r_c} B' \right) EI_x e_0 \right]' + \frac{p_{16}}{r_c^2} EI_z (\Lambda e_0)' \\
& + \frac{1}{r_c^2} \left\{ \left[p_9 \Lambda + p_{16} \left(\frac{D}{r_s} - \frac{2}{r_c} \xi \right) \right] EI_{xz} e_0 \right\}' + \frac{E\Gamma}{r_s r_c} b_{20} (b_{20} \phi'''' + b_{21} u'''') - \frac{GJ}{r_c} r_s b_{20} (b_{20} \phi'' + b_{21} u'') \\
& - p \frac{\bar{A}_r}{r_c} \left(-b_1 + \frac{b_2}{r_c} B'' - \frac{p_2}{r_c} \xi' \right) \tag{D1d}
\end{aligned}$$

$$\begin{aligned}
Q_5 = & \frac{b_{17}}{r_c^2} \xi \left(N - \frac{M_x}{r_c} \right) - \frac{b_{17}}{r_c} \left(N - \frac{M_x}{r_c} \right)' - \frac{p_{12}}{r_c} M_x' + \frac{1}{r_c^2} \left(-\frac{b_{17}}{r_s} D + p_{15} \xi \right) \left(M_x - \frac{EI_x e_0}{r_c} \right) - \frac{b_{18}}{r_c} M_z + \frac{1}{r_c^2} (b_{18} \xi - b_{17} \Lambda) \left(M_z \right. \\
& \left. - \frac{EI_{xz}}{r_c} e_0 \right) + \frac{1}{r_c^2} \left(\frac{p_{23}}{r_c} \xi - \frac{p_{15}}{r_s} D \right) EI_x e_0 - \frac{b_{18}}{r_c^2} EI_z \Lambda e_0 - \frac{1}{r_c^2} \left(p_{15} \Lambda + \frac{b_{18}}{r_s} D - \frac{2}{r_c} b_{18} \xi \right) EI_{xz} e_0 + A_r G \gamma_{yx} \\
& - p \bar{A}_r \left(-\gamma_{yx} + \frac{b_{17}}{r_c^2} \xi \right) \tag{D1e}
\end{aligned}$$

$$\begin{aligned}
Q_6 = & \frac{b_{15}}{r_c^2} \xi \left(N - \frac{M_x}{r_c} \right) - \frac{b_{15}}{r_c} \left(N - \frac{M_x}{r_c} \right)' - \frac{p_{11}}{r_c} M_x' + \frac{1}{r_c^2} \left(-\frac{b_{15}}{r_c} D + p_{14} \xi \right) \left(M_x - \frac{EI_x e_0}{r_c} \right) - \frac{b_{15}}{r_c^2} \Lambda \left(M_z - \frac{EI_{xz}}{r_c} e_0 \right) + \frac{1}{r_c^2} \left(\frac{p_{22}}{r_c} \xi \right. \\
& \left. - \frac{p_{14}}{r_s} D \right) EI_x e_0 - \frac{p_{14}}{r_c^2} \Lambda EI_{xz} e_0 + A_r G \gamma_{yz} - p \bar{A}_r \left(-\gamma_{yz} + \frac{b_{15}}{r_c^2} \xi \right) \tag{D1f}
\end{aligned}$$

APPENDIX D – Continued

where

$$p_1 = b_5 - 1$$

$$p_2 = b_1 + b_7$$

$$p_3 = 0$$

$$p_4 = -b_4$$

$$p_5 = \frac{b_5 - 1}{r_c}$$

$$p_6 = b_8 + \frac{b_7}{r_c}$$

$$p_7 = \frac{1}{r_c}$$

$$p_8 = 1 - \frac{2b_2}{r_c}$$

$$p_9 = b_8 + \frac{2(b_1 + b_7)}{r_c}$$

$$p_{10} = \frac{x_c}{r_c r_s}$$

$$p_{11} = -\frac{r_s}{r_c}$$

$$p_{12} = -\frac{x_c}{r_c}$$

$$p_{13} = \frac{x_c}{r_c r_s} + \frac{b_{10}}{r_c}$$

$$p_{14} = \frac{b_{15} - r_s}{r_c}$$

$$p_{15} = \frac{b_{17} - x_c}{r_c}$$

$$p_{16} = b_9 - 1$$

$$p_{17} = -b_4$$

$$p_{18} = \frac{b_5 - 1}{r_c}$$

$$p_{19} = \frac{2x_c}{r_c r_s} + \frac{b_{10}}{r_c}$$

$$p_{20} = 2b_8 + \frac{3(b_1 + b_7)}{r_c}$$

$$p_{21} = 2 - \frac{3b_2}{r_c}$$

$$p_{22} = \frac{b_{15} - 2r_s}{r_c}$$

$$p_{23} = \frac{b_{17} - 2x_c}{r_c}$$

(D2)

and

$$\left. \begin{aligned} \xi &= C + A' \\ \Lambda &= \frac{F}{r_s} + \phi' \end{aligned} \right\}$$

(D3)

The terms b_1, b_2, \dots, b_{21} are given by equations (B5) and (C14); the terms $A, B, C, D,$ and F are given by equations (3).

APPENDIX D – Continued

The strain, hoop force, and moments in the ring for the initial equilibrium state are

$$\left. \begin{aligned}
 e_{0,A} &= -\frac{A_A}{r_c} \\
 N_A &= \frac{E}{r_c} \left[-A_A \left(A_r + \frac{I_x}{r_c^2} \right) + \phi_A \frac{I_{xz}}{r_c} \right] \\
 M_{x,A} &= \frac{E}{r_c} \left(-A_A \frac{I_x}{r_c} + \phi_A I_{xz} \right) \\
 M_{z,A} &= \frac{E}{r_c} \left(-A_A \frac{I_{xz}}{r_c} + \phi_A I_z \right)
 \end{aligned} \right\} \quad (D4)$$

The hoop force and moments in the ring for the incremental equilibrium state are

$$\left. \begin{aligned}
 N_B &= K_1 v'_B + K_2 w''_B + K_3 u''_B + K_4 \phi''_B + K_5 w_B + K_6 \phi_B + K_7 \gamma'_{yz,B} + K_8 \gamma'_{yx,B} \\
 M_{x,B} &= K_9 v'_B + K_{10} w''_B + K_{11} u''_B + K_{12} \phi''_B + K_{13} w_B + K_{14} \phi_B + K_{15} \gamma'_{yz,B} + K_{16} \gamma'_{yx,B} \\
 M_{z,B} &= K_{17} v'_B + K_{18} w''_B + K_{19} u''_B + K_{20} \phi''_B + K_{21} w_B + K_{22} \phi_B + K_{23} \gamma'_{yz,B} + K_{24} \gamma'_{yx,B}
 \end{aligned} \right\} \quad (D5)$$

where the K_i are given by

APPENDIX D – Concluded

$$K_1 = \frac{EA_r}{r_c} b_3$$

$$K_{13} = \frac{EI_x}{r_c^2}$$

$$K_2 = \frac{EA_r}{r_c} b_5 + \frac{EI_x}{r_c^3}$$

$$K_{14} = -\frac{EI_x}{r_c^2} b_1 + \frac{EI_{xz}}{r_c}$$

$$K_3 = \frac{EA_r}{r_c} b_{10} + \frac{EI_x}{r_c^3 r_s} x_c + \frac{EI_{xz}}{r_c^2} b_{12}$$

$$K_{15} = -\frac{EI_x}{r_c^2} r_s$$

$$K_4 = \frac{EA_r}{r_c} b_7 + \frac{EI_x}{r_c^3} (b_7 + r_c b_8) + \frac{EI_{xz}}{r_c^2} b_9$$

$$K_{16} = -\frac{EI_x}{r_c^2} x_c - \frac{EI_{xz}}{r_c}$$

$$K_5 = \frac{EA_r}{r_c} + \frac{EI_x}{r_c^3}$$

$$K_{17} = 0$$

$$K_6 = -\frac{EA_r}{r_c} b_1 - \frac{EI_x}{r_c^3} b_1 + \frac{EI_{xz}}{r_c^2}$$

$$K_{18} = \frac{EI_{xz}}{r_c^2}$$

$$K_7 = \frac{EA_r}{r_c} b_{15} - \frac{EI_x}{r_c^3} r_s$$

$$K_{19} = \frac{EI_{xz}}{r_c^2 r_s} x_c + \frac{EI_z}{r_c} b_{12}$$

$$K_8 = \frac{EA_r}{r_c} b_{17} - \frac{EI_x}{r_c^3} x_c - \frac{EI_{xz}}{r_c^2}$$

$$K_{20} = \frac{EI_{xz}}{r_c^2} (b_7 + r_c b_8) + \frac{EI_z}{r_c} b_9$$

$$K_9 = 0$$

$$K_{21} = \frac{EI_{xz}}{r_c^2}$$

$$K_{10} = \frac{EI_x}{r_c^2}$$

$$K_{22} = -\frac{EI_{xz}}{r_c^2} b_1 + \frac{EI_z}{r_c}$$

$$K_{11} = \frac{EI_x}{r_c^2 r_s} x_c + \frac{EI_{xz}}{r_c} b_{12}$$

$$K_{23} = -\frac{EI_{xz}}{r_c^2} r_s$$

$$K_{12} = \frac{EI_x}{r_c^2} (b_7 + r_c b_8) + \frac{EI_{xz}}{r_c} b_9$$

$$K_{24} = -\frac{EI_{xz}}{r_c^2} x_c - \frac{EI_z}{r_c}$$

(D6)

APPENDIX E

COEFFICIENTS OF RING STIFFNESS MATRIX

A listing is provided of the coefficients for the ring stiffness matrix which appears in equations (12). The coefficients which are a function of the initial state A are

$$\bar{G}_{11,A} = n^2 \left[\frac{N_A}{r_c^2} (b_{10}^2 + 1) + \frac{M_{x,A}}{r_c^3} \left(1 - b_{10}^2 + 2b_{10} \frac{x_c}{r_s} \right) + \frac{2M_{z,A}}{r_c^2 r_s} b_{10} + \frac{EI_{xO,A}}{r_c^4} \left(\frac{x_c^2}{r_s^2} + 1 \right) + \frac{2EI_{xzO,A}}{r_c^3 r_s^2} x_c + \frac{EI_{zO,A}}{r_c^2 r_s^2} \right] \quad (E1a)$$

$$\bar{G}_{12,A} = -\frac{nb_3}{r_c^2} \left(N_A b_{10} + M_{x,A} b_{11} + \frac{M_{z,A}}{r_s} \right) \quad (E1b)$$

$$\bar{G}_{13,A} = n^2 \left\{ N_A \frac{b_{10}(b_5 - 1)}{r_c^2} + \frac{M_{x,A}}{r_c^2} \left[b_6 b_{10} + \frac{(b_5 - 1)x_c}{r_c r_s} \right] + \frac{M_{z,A}}{r_c^2 r_s} (b_5 - 1) \right\} \quad (E1c)$$

$$\begin{aligned} \bar{G}_{14,A} = n^2 \left\{ \frac{N_A}{r_c^2} \left[(b_1 + b_7) b_{10} + b_2 \right] + \frac{M_{x,A}}{r_c^2} \left[\frac{x_c}{r_c r_s} (b_1 + b_7) + b_8 b_{10} + \frac{b_2}{r_c} - 1 \right] + \frac{M_{z,A}}{r_c^2} \left[\frac{(b_1 + b_7)}{r_s} + (b_9 - 1) b_{10} \right] \right. \\ \left. + \frac{EI_{xO,A}}{r_c^4} \left[\frac{x_c}{r_s} (b_1 + b_7 + r_c b_8) + b_2 - r_c \right] + \frac{EI_{xzO,A}}{r_c^3 r_s} \left[b_1 + b_7 + r_c b_8 + x_c (b_9 - 1) \right] + \frac{EI_{zO,A}}{r_c^2 r_s} (b_9 - 1) \right\} \quad (E1d) \end{aligned}$$

$$\bar{G}_{15,A} = n \left[-N_A \frac{b_{10} b_{15}}{r_c^2} + \frac{M_{x,A}}{r_c^3} (b_{10} r_s - r_c b_{11} b_{15}) - M_{z,A} \frac{b_{15}}{r_c^2 r_s} + \frac{EI_{xO,A}}{r_c^4} x_c + \frac{EI_{xzO,A}}{r_c^3} \right] \quad (E1e)$$

$$\bar{G}_{16,A} = n \left[-N_A \frac{b_{10}b_{17}}{r_c^2} + \frac{M_{x,A}}{r_c^3} (b_{10}x_c - r_c b_{11}b_{17}) + \frac{M_{z,A}}{r_c^2} \left(-\frac{b_{17}}{r_s} + b_{10} \right) + \frac{EI_{x^2}e_{0,A}}{r_c^4 r_s} x_c^2 + \frac{2EI_{xz}e_{0,A}}{r_c^3 r_s} x_c + \frac{EI_z e_{0,A}}{r_c^2 r_s} \right] \quad (E1f)$$

$$\bar{G}_{22,A} = \frac{b_3}{r_c^2} \left(N_A - \frac{M_{x,A}}{r_c} \right) \quad (E1g)$$

$$\bar{G}_{23,A} = \frac{nb_3}{r_c^2} \left[-N_A (b_5 - 1) - M_{x,A} b_6 \right] \quad (E1h)$$

$$\bar{G}_{24,A} = \frac{nb_3}{r_c^2} \left[-N_A (b_1 + b_7) - M_{x,A} b_8 - M_{z,A} (b_9 - 1) \right] \quad (E1i)$$

$$\bar{G}_{25,A} = \frac{b_3}{r_c^2} \left(N_A b_{15} + M_{x,A} b_{16} \right) \quad (E1j)$$

$$\bar{G}_{26,A} = \frac{b_3}{r_c^2} \left(N_A b_{17} + M_{x,A} b_{19} - M_{z,A} \right) \quad (E1k)$$

$$\bar{G}_{33,A} = n^2 \left[N_A \frac{(b_5 - 1)^2}{r_c^2} + \frac{M_{x,A}}{r_c^2} (b_5 - 1) b_6 \right] \quad (E1l)$$

$$\bar{G}_{34,A} = n^2 \left[N_A \frac{(b_1 + b_7)(b_5 - 1)}{r_c^2} + \frac{M_{x,A}}{r_c^2} b_8 (b_5 - 1) + \frac{M_{z,A}}{r_c^2} (b_9 - 1)(b_5 - 1) \right] \quad (E1m)$$

$$\bar{G}_{35,A} = -n \left\{ N_A \frac{(b_5 - 1)b_{15}}{r_c^2} + \frac{M_{x,A}}{r_c^3} \left[r_c b_6 b_{15} - r_s (b_5 - 1) \right] \right\} \quad (E1n)$$

$$\bar{G}_{36,A} = -n \left\{ N_A \frac{(b_5 - 1)b_{17}}{r_c^2} + \frac{M_{x,A}}{r_c^3} \left[r_c b_6 b_{17} - x_c (b_5 - 1) \right] - \frac{M_{z,A}}{r_c^2} (b_5 - 1) \right\} \quad (E1o)$$

$$\begin{aligned} \bar{G}_{44,A} = n^2 \left\{ N_A \frac{(b_1 + b_7)^2 + b_2^2}{r_c^2} + \frac{M_{x,A}}{r_c^2} \left[(b_1 + b_7) \left(\frac{b_1 + b_7 + 2r_c b_8}{r_c} \right) - b_2 \left(2 - \frac{b_2}{r_c} \right) \right] + \frac{2M_{z,A}}{r_c^2} (b_9 - 1) (b_1 + b_7) \right. \\ \left. + \frac{EI_x e_{o,A}}{r_c^3} \left[\frac{b_1 + b_7}{r_c} (b_1 + b_7 + 2r_c b_8) + r_c b_8^2 - 2b_2 + \frac{b_2^2}{r_c} + r_c \right] + \frac{2EI_{xz} e_{o,A}}{r_c^3} (b_9 - 1) (b_1 + b_7 + r_c b_8) \right. \\ \left. + \frac{EI_z e_{o,A}}{r_c^2} (b_9 - 1)^2 \right\} \quad (E1p) \end{aligned}$$

$$\begin{aligned} \bar{G}_{45,A} = -n \left\{ N_A \frac{(b_1 + b_7)b_{15}}{r_c^2} + \frac{M_{x,A}}{r_c^3} \left[-r_s (b_1 + b_7) + r_c b_8 b_{15} \right] + \frac{M_{z,A}}{r_c^2} b_{15} (b_9 - 1) - \frac{EI_x e_{o,A}}{r_c^4} r_s (b_1 + b_7 + r_c b_8) \right. \\ \left. - \frac{EI_{xz} e_{o,A}}{r_c^3} r_s (b_9 - 1) \right\} \quad (E1q) \end{aligned}$$

$$\begin{aligned} \bar{G}_{46,A} = -n \left\{ N_A \frac{(b_1 + b_7)b_{17}}{r_c^2} + \frac{M_{x,A}}{r_c^3} \left[r_c b_8 b_{17} - x_c (b_1 + b_7) \right] + \frac{M_{z,A}}{r_c^2} \left[b_{17}(b_9 - 1) - (b_1 + b_7) \right] \right. \\ \left. - \frac{EI_x e_{o,A}}{r_c^4} x_c (b_1 + b_7 + r_c b_8) - \frac{EI_{xz} e_{o,A}}{r_c^3} \left[(b_9 - 1)x_c + b_1 + b_7 + r_c b_8 \right] - \frac{EI_z e_{o,A}}{r_c^2} (b_9 - 1) \right\} \end{aligned} \quad (E1r)$$

$$\bar{G}_{55,A} = N_A \frac{b_{15}^2}{r_c^2} + \frac{M_{x,A}}{r_c^3} b_{15} (-r_s + r_c b_{16}) + \frac{EI_x e_{o,A}}{r_c^4} r_s^2 \quad (E1s)$$

$$\bar{G}_{56,A} = N_A \frac{b_{15} b_{17}}{r_c^2} + \frac{M_{x,A}}{r_c^3} (-r_s b_{17} + r_c b_{15} b_{19}) - \frac{M_{z,A} b_{15}}{r_c^2} + \frac{EI_x e_{o,A}}{r_c^4} x_c r_s + \frac{EI_{xz} e_{o,A}}{r_c^3} r_s \quad (E1t)$$

$$\bar{G}_{66,A} = N_A \frac{b_{17}^2}{r_c^2} + \frac{M_{x,A}}{r_c^3} b_{17} (-b_{17} - 2x_c) - \frac{2M_{z,A} b_{17}}{r_c^2} + \frac{EI_x e_{o,A}}{r_c^4} x_c^2 - \frac{2EI_{xz} e_{o,A}}{r_c^3} (b_{17} - x_c) + \frac{EI_z e_{o,A}}{r_c^2} \quad (E1u)$$

The prestress quantities appearing in equations (E1) are defined in equations (D4) and the constants b_i are defined in equations (B5) and (C14).

The coefficients which are a function of the incremental state B are:

$$\bar{G}_{11,B} = n^2 \left[-\frac{p\bar{A}_r}{r_c^2} (b_{10}^2 + 1) + \frac{GJ}{r_c r_s^3} + n^2 \left(\frac{EA_r b_{10}^2}{r_c^2} + \frac{EI_x x_c^2}{r_c^4 r_s^2} + \frac{2EI_{xz} x_c}{r_c^3 r_s^2} + \frac{EI_z}{r_c^2 r_s^2} + \frac{E\Gamma}{r_c r_s^5} \right) \right] \quad (E2a)$$

$$\bar{G}_{12,B} = n \frac{b_3 b_{10}}{r_c^2} (p\bar{A}_R - n^2 EA_R) \quad (\text{E2b})$$

$$\bar{G}_{13,B} = -n^2 \left[\frac{p\bar{A}_R}{r_c^2} (b_5 - 1) b_{10} + \frac{EA_R b_{10}}{r_c^2} + \frac{EI_x x_c}{r_c^4 r_s} + \frac{EI_{xz}}{r_c^3 r_s} \right] + n^4 \left(\frac{EA_R}{r_c^2} b_5 b_{10} + \frac{EI_x x_c}{r_c^4 r_s} + \frac{EI_{xz}}{r_c^3 r_s} \right) \quad (\text{E2c})$$

$$\bar{G}_{14,B} = n^2 \left\{ -\frac{p\bar{A}_R}{r_c^2} \left[b_{10} (b_1 + b_7) + b_2 \right] + \frac{EA_R}{r_c^2} b_1 b_{10} + \frac{EI_x}{r_c^4 r_s} b_1 x_c + \frac{EI_{xz}}{r_c^3 r_s} (b_1 - x_c) - \frac{EI_z}{r_c^2 r_s} + \frac{GJ}{r_c r_s^2} (b_5 - 1) \right\} \\ + n^4 \left[\frac{EA_R}{r_c^2} b_7 b_{10} + \frac{EI_x}{r_c^4 r_s} x_c (b_7 + r_c b_8) + \frac{EI_{xz}}{r_c^3 r_s} (b_9 x_c + b_7 + r_c b_8) + \frac{EI_z b_9}{r_c^2 r_s} + \frac{E\Gamma}{r_c r_s^4} (b_9 - 1) \right] \quad (\text{E2d})$$

$$\bar{G}_{15,B} = n \frac{p\bar{A}_R}{r_c^2} b_{10} b_{15} - n^3 \left(\frac{EA_R}{r_c^2} b_{10} b_{15} - \frac{EI_x}{r_c^4} x_c - \frac{EI_{xz}}{r_c^3} \right) \quad (\text{E2e})$$

$$\bar{G}_{16,B} = n \frac{p\bar{A}_R}{r_c^2} b_{10} b_{17} - n^3 \left(\frac{EA_R}{r_c^2} b_{10} b_{17} - \frac{EI_x}{r_c^4 r_s} x_c^2 - \frac{2EI_{xz}}{r_c^3 r_s} x_c - \frac{EI_z}{r_c^2 r_s} \right) \quad (\text{E2f})$$

$$\bar{G}_{22,B} = \frac{b_3^2}{r_c^2} (-p\bar{A}_R + n^2 EA_R) \quad (\text{E2g})$$

$$\bar{G}_{23,B} = n \frac{b_3}{r_c^2} \left[p\bar{A}_R (b_5 - 1) + EA_R - n^2 EA_R b_5 \right] \quad (\text{E2h})$$

$$\bar{G}_{24,B} = n \frac{b_3}{r_c^2} \left[p\bar{A}_r (b_1 + b_7) - EA_r b_1 \right] - n^3 \frac{EA_r}{r_c^2} b_3 b_7 \quad (E2i)$$

$$\bar{G}_{25,B} = \frac{b_3 b_{15}}{r_c^2} \left(-p\bar{A}_r + n^2 EA_r \right) \quad (E2j)$$

$$\bar{G}_{26,B} = -\frac{p\bar{A}_r}{r_c^2} b_3 b_{17} + n^2 \frac{EA_r}{r_c^2} b_3 b_{17} \quad (E2k)$$

$$\bar{G}_{33,B} = \frac{EA_r}{r_c^2} + \frac{EI_x}{r_c^4} - n^2 \left[\frac{p\bar{A}_r}{r_c} (b_5 - 1)^2 + \frac{2EA_r}{r_c^2} b_5 + \frac{2EI_x}{r_c^4} \right] + n^4 \left(\frac{EA_r}{r_c^2} b_5^2 + \frac{EI_x}{r_c^4} \right) \quad (E2l)$$

$$\bar{G}_{34,B} = -\frac{EA_r}{r_c^2} b_1 - \frac{EI_x}{r_c^4} b_1 + \frac{EI_{xz}}{r_c^3} - n^2 \left[\frac{p\bar{A}_r}{r_c^2} (b_5 - 1) (b_1 + b_7) + \frac{EA_r}{r_c^2} (b_7 - b_1 b_5) + \frac{EI_x}{r_c^4} (-b_1 + b_7 + r_c b_8) + \frac{EI_{xz}}{r_c^3} (1 + b_9) \right] \\ + n^4 \left[\frac{EA_r}{r_c^2} b_5 b_7 + \frac{EI_x}{r_c^4} (b_7 + r_c b_8) + \frac{EI_{xz}}{r_c^3} b_9 \right] \quad (E2m)$$

$$\bar{G}_{35,B} = n \left[\frac{p\bar{A}_r}{r_c^2} (b_5 - 1) b_{15} + \frac{EA_r}{r_c^2} b_{15} - \frac{EI_x r_s}{r_c^4} \right] - n^3 \left(\frac{EA_r}{r_c^2} b_5 b_{15} - \frac{EI_x r_s}{r_c^4} \right) \quad (E2n)$$

$$\bar{G}_{36,B} = n \left[\frac{p\bar{A}_r}{r_c^2} (b_5 - 1) b_{17} + \frac{EA_r}{r_c^2} b_{17} - \frac{EI_x}{r_c^4} x_c - \frac{EI_{xz}}{r_c^3} \right] - n^3 \left(\frac{EA_r}{r_c^2} b_5 b_{17} - \frac{EI_x}{r_c^4} x_c - \frac{EI_{xz}}{r_c^3} \right) \quad (E2o)$$

$$\begin{aligned} \bar{G}_{44,B} = & \frac{EA_r}{r_c^2} b_1^2 + \frac{EI_x}{r_c^4} b_1^2 - \frac{2EI_{xz}}{r_c^3} b_1 + \frac{EI_z}{r_c^2} + n^2 \left\{ -\frac{p\bar{A}_r}{r_c^2} \left[b_2^2 + (b_1 + b_7)^2 \right] + \frac{2EA_r}{r_c^2} b_1 b_7 + \frac{2EI_x}{r_c^4} b_1 (b_7 + r_c b_8) \right. \\ & + \frac{2EI_{xz}}{r_c^3} (b_1 b_9 - b_7 - r_c b_8) - \frac{2EI_z}{r_c^2} b_9 + \frac{GJ}{r_c r_s} (1 - b_9)^2 \left. \right\} + n^4 \left[\frac{EA_r}{r_c^2} b_7^2 + \frac{EI_x}{r_c^4} (b_7 + r_c b_8)^2 + \frac{2EI_{xz}}{r_c^3} b_9 (b_7 + r_c b_8) \right. \\ & \left. + \frac{EI_z}{r_c^2} b_9^2 + \frac{E\Gamma}{r_c r_s^3} (1 - b_9)^2 \right] \quad (E2p) \end{aligned}$$

$$\bar{G}_{45,B} = n \left[\frac{p\bar{A}_r}{r_c^2} b_{15} (b_1 + b_7) - \frac{EA_r}{r_c^2} b_1 b_{15} + \frac{EI_x}{r_c^4} r_s b_1 - \frac{EI_{xz}}{r_c^3} r_s \right] + n^3 \left[-\frac{EA_r}{r_c^2} b_7 b_{15} + \frac{EI_x}{r_c^4} r_s (b_7 + r_c b_8) + \frac{EI_{xz}}{r_c^3} r_s b_9 \right] \quad (E2q)$$

$$\begin{aligned} \bar{G}_{46,B} = & n \left[\frac{p\bar{A}_r}{r_c^2} (b_1 + b_7) b_{17} - \frac{EA_r}{r_c^2} b_1 b_{17} + \frac{EI_x}{r_c^4} b_1 x_c - \frac{EI_{xz}}{r_c^3} (-b_1 + x_c) - \frac{EI_z}{r_c^2} \right] + n^3 \left[-\frac{EA_r}{r_c^2} b_7 b_{17} + \frac{EI_x}{r_c^4} x_c (b_7 + r_c b_8) \right. \\ & \left. + \frac{EI_{xz}}{r_c^3} (b_9 x_c + b_7 + r_c b_8) + \frac{EI_z}{r_c^2} b_9 \right] \quad (E2r) \end{aligned}$$

$$\bar{G}_{55,B} = GA_r + p\bar{A}_r \left(1 - \frac{b_{15}^2}{r_c^2} \right) + n^2 \left(\frac{EA_r}{r_c^2} b_{15}^2 + \frac{EI_x}{r_c^4} r_s^2 \right) \quad (E2s)$$

APPENDIX E - Concluded

$$\bar{G}_{56,B} = -\frac{p\bar{A}_r}{r_c^2} b_{15}b_{17} + n^2 \left(\frac{EA_r}{r_c^2} b_{15}b_{17} + \frac{EI_x}{r_c^4} r_s x_c + \frac{EI_{xz}}{r_c^3} r_s \right) \quad (E2t)$$

$$\bar{G}_{66,B} = GA_r + p\bar{A}_r \left(1 - \frac{b_{17}^2}{r_c^2} \right) + n^2 \left(\frac{EA_r}{r_c^2} b_{17}^2 + \frac{EI_x}{r_c^4} x_c^2 + \frac{2EI_{xz}}{r_c^3} x_c + \frac{EI_z}{r_c^2} \right) \quad (E2u)$$

APPENDIX F

FORMULATION OF BOUNDARY CONDITIONS IN INTRINSIC COORDINATE SYSTEM

The boundary conditions given by equations (1) are expressed in a cylindrical coordinate system and the shell variables are also in that coordinate system. However, the boundary conditions are usually difficult to use in this form in connection with shell programs that utilize normal and tangential displacements as variables. A more convenient form is to express these boundary conditions in the cylindrical coordinate system in terms of the shell variables in the intrinsic coordinate system.

From figure F1, it is noted that the stress resultants and middle-surface displacements in the cylindrical coordinate system are related to those in the intrinsic coordinate

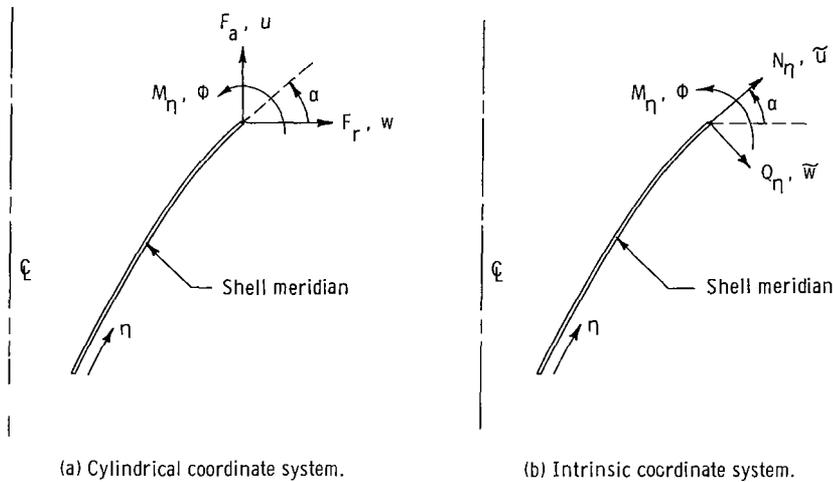


Figure F1.- Displacements and stress resultants in cylindrical and intrinsic coordinate systems.

system by the following equations:

$$\left. \begin{aligned}
 F_a &= -Q_\eta \cos \alpha + N_\eta \sin \alpha \\
 F_r &= N_\eta \cos \alpha + Q_\eta \sin \alpha \\
 u &= \tilde{u} \sin \alpha - \tilde{w} \cos \alpha \\
 w &= \tilde{w} \sin \alpha + \tilde{u} \cos \alpha
 \end{aligned} \right\} \quad (F1)$$

The circumferential displacement and rotation, as well as the stress and moment resultants F_θ and M_η , are the same in both coordinate systems.

By perturbing equations (F1), by subtracting out the initial equilibrium state (subscript A), and in addition by defining

$$\left. \begin{aligned} Q_{\eta,B} &= Q_{\eta,B}^{(n)} \cos n\theta \\ N_{\eta,B} &= N_{\eta,B}^{(n)} \cos n\theta \\ \tilde{u}_B &= \tilde{u}_B^{(n)} \cos n\theta \\ \tilde{w}_B &= \tilde{w}_B^{(n)} \cos n\theta \end{aligned} \right\} \quad (F2)$$

the boundary conditions in the cylindrical coordinate system in terms of the shell variables in the intrinsic coordinate system can be written as

$$\mathbf{r} \begin{bmatrix} \sin \alpha & 0 & -\cos \alpha & 0 \\ 0 & 1 & 0 & 0 \\ \cos \alpha & 0 & \sin \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} N_{\eta,B}^{(n)} \\ F_{\theta,B}^{(n)} \\ Q_{\eta,B}^{(n)} \\ M_{\eta,B}^{(n)} \end{Bmatrix} \pm \begin{bmatrix} G_{11} \sin \alpha + G_{13} \cos \alpha & G_{12} & -G_{11} \cos \alpha + G_{13} \sin \alpha & G_{14} \\ G_{21} \sin \alpha + G_{23} \cos \alpha & G_{22} & -G_{21} \cos \alpha + G_{23} \sin \alpha & G_{24} \\ G_{31} \sin \alpha + G_{33} \cos \alpha & G_{32} & -G_{31} \cos \alpha + G_{33} \sin \alpha & G_{34} \\ G_{41} \sin \alpha + G_{43} \cos \alpha & G_{42} & -G_{41} \cos \alpha + G_{43} \sin \alpha & G_{44} \end{bmatrix} \begin{Bmatrix} \tilde{u}_B^{(n)} \\ v_B^{(n)} \\ \tilde{w}_B^{(n)} \\ \phi_B^{(n)} \end{Bmatrix} = \begin{Bmatrix} \zeta_{1,B}^{(n)} \\ \zeta_{2,B}^{(n)} \\ \zeta_{3,B}^{(n)} \\ \zeta_{4,B}^{(n)} \end{Bmatrix} \quad (F3)$$

In some instances, it may be advantageous to express the boundary conditions in the intrinsic coordinate system and in terms of the shell variables in that intrinsic coordinate system. By eliminating the coupling terms in the coefficient of the force vector in equations (F3), the following new form for the boundary conditions is obtained:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} N_{\eta,B}^{(n)} \\ F_{\theta,B}^{(n)} \\ Q_{\eta,B}^{(n)} \\ M_{\eta,B}^{(n)} \end{Bmatrix} \pm \begin{bmatrix} G_{11} \sin^2 \alpha + 2G_{13} \sin \alpha \cos \alpha + G_{33} \cos^2 \alpha & G_{12} \sin \alpha + G_{32} \cos \alpha & (-G_{11} + G_{33}) \sin \alpha \cos \alpha + G_{13} (\sin^2 \alpha - \cos^2 \alpha) & G_{14} \sin \alpha + G_{34} \cos \alpha \\ \text{Symmetric} & G_{22} & -G_{21} \cos \alpha + G_{23} \sin \alpha & G_{24} \\ G_{11} \cos^2 \alpha - 2G_{13} \sin \alpha \cos \alpha + G_{33} \sin^2 \alpha & -G_{14} \cos \alpha + G_{34} \sin \alpha & & G_{44} \end{bmatrix} \begin{Bmatrix} \bar{u}_B^{(n)} \\ v_B^{(n)} \\ \bar{w}_B^{(n)} \\ \phi_B^{(n)} \end{Bmatrix} = \begin{Bmatrix} \lambda_{1,B}^{(n)} \sin \alpha + \lambda_{3,B}^{(n)} \cos \alpha \\ \lambda_{2,B}^{(n)} \\ -\lambda_{1,B}^{(n)} \cos \alpha + \lambda_{3,B}^{(n)} \sin \alpha \\ \lambda_{4,B}^{(n)} \end{Bmatrix} \quad (\text{F4})$$

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